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Developing Understanding for Different Roles of Proof in Dynamic Geometry

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Introduction

In a recent article submitted to *Philosophae Mathematicae* Yehuda Rav (1999) poses the interesting hypothetical situation of us having access to an all-powerful computer called PYTHIAGORA with which we can quickly check whether any conceivable mathematical conjecture is true or not. Would such a powerful tool spell the end of proof as we know it today?

Perhaps surprisingly to the general public, the answer to this question is a resounding "*NO!*" As Rav points out, it is quite often irrelevant in mathematics whether a particular conjecture is true or not. He gives the example of the still unproved Goldbach conjecture that has been the fundamental catalyst for the development of major new theories as mathematicians search for a proof:

> "*Look at the treasure which attempted proofs of the Goldbach conjecture has produced, and how much less significant by comparison its ultimate 'truth value' might be! ... Now let us suppose that one day somebody comes up with a counter-example to the Goldbach conjecture or with a proof that there exist positive even integers not representable as a sum of two primes. Would that falsify or just tarnish all the magnificent theories, concepts and techniques which were developed in order to prove the now supposed incorrect conjecture? None of that. A disproof of the Goldbach conjecture would just catalyze a host of* ***new*** *developments, without the slightest effect on hitherto developed* ***methods*** *in an attempt to prove the conjecture. For we would immediately ask new questions, such as to the number of 'non-goldbachian' even integers: finitely many? infinitely many? ... New treasures would be accumulated alongside, rather than instead of the old ones - thus and so is the path of proofs in mathematics!*"

A little further on Yehuda Rav emphasizes that in fact proofs rather than theorems are the bearers of mathematical knowledge:

> "*Theorems are in a sense just tags, labels for proofs, summaries of information, headlines of news, editorial devices. The whole arsenal of mathematical methodologies, concepts, strategies and techniques for solving*

> *problems, the establishment of interconnections between theories, the systematization of results - the entire mathematical know-how is embedded in proofs. ....Think of proofs as a network of roads in a public transportation system, and regard statements of theorems as bus stops; the site of the stops is just a matter of convenience.*"

In a similar vein, the research mathematician Gian-Carlo Rota (1997:190) pointed out, regarding the recent proof of Fermat's Last Theorem, that the value of the proof goes far beyond that of mere verification of the result:

> "*The actual value of what Wiles and his collaborators did is far greater than the mere proof of a whimsical conjecture. The point of the proof of Fermat's last theorem is to open up new possibilities for mathematics. ... The value of Wiles's proof lies not in what it proves, but in what it opens up, in what it makes possible.*"

Several years ago the mathematician Paul Halmos in Albers (1982:239-240) similarly pointed out that although the computer-aided proof by Appel and Haken of the four-color conjecture in 1976 convinced him that it was true, this gave no deeper insight or understanding into why it was true:

> "*... I am much less likely now, after their work, to go looking for a counter-example to the four-color conjecture than I was before. To that extent, what has happened convinced me that the four-color theorem is true. I have a religious belief that some day soon, maybe six months from now, maybe sixty years from now, somebody will write a proof of the four-color theorem that will take up sixty pages in the Pacific Journal of Mathematics. Soon after that, perhaps six months or sixty years later, somebody will write a four-page proof, based on the concepts that in the meantime we will have developed and studied and understood. The result will belong to the grand, glorious, architectural structure of mathematics... mathematics isn't in a hurry. Efficiency is meaningless. Understanding is what counts*."

Two important ideas that clearly emanate from the above quotes are, first, that proofs are an indispensable part of mathematical knowledge, and second, that their value goes far beyond the mere verification of results. The first idea refutes the growing public misconception that powerful new computer tools like *Sketchpad*, *Mathematica*, etc. are making proof obsolete (see for example Horgan, 1993). Although such tools enable us to gain conviction through visualization or empirical measurement, proofs are still as important as ever. In addition, as alluded to in the second idea above, proofs are also extremely valuable as they can provide insights, lead to new discoveries or assist systematization. These multiple roles of proof are the main ideas that will be explored a little further in this paper.

The Role and Function of Proof

Traditionally the function of proof has been seen almost exclusively in terms of the **verification** (conviction or justification) of the correctness of mathematical statements. The idea is that proof is used mainly to remove either personal doubt and/or those of skeptics; an idea which has one-sidedly dominated teaching practice and most discussions and research on the teaching of proof. For instance, according to Kline (1973:151): "*a proof is only meaningful when it answers the student's **doubts**, when it proves what is not obvious.*" (bold added).

However, proof has many other important functions within mathematics, which in some situations are of far greater importance to mathematicians than that of mere verification. Some of these are (compare De Villiers, 1997; 1998; 1999; 2001):

- explanation (providing insight into why it is true)
- discovery (the discovery or invention of new results)
- communication (the negotiation of meaning)
- intellectual challenge (the self-realization/fulfilment derived from constructing a proof)
- systematisation (the organisation of various results into a deductive system of axioms, concepts and theorems)

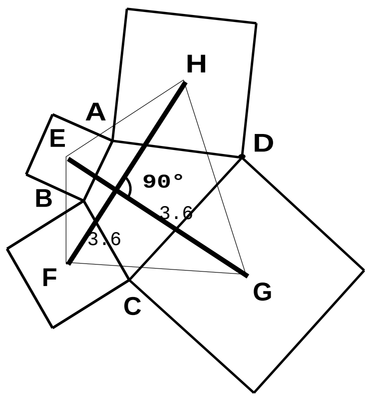

Figure 1

Proof as a means of explanation (illumination)

Although many mathematics teachers seem to believe that proof is an absolute prerequisite to conviction, in actual mathematical practice, conviction is probably far more frequently a prerequisite for the finding of a proof. For example, some years ago I came across Van Aubel's theorem in Gardner (1981:176-179), namely, that the centers of squares on the sides of any quadrilateral *ABCD*, form a quadrilateral *EFGH* with equal and perpendicular diagonals (see Figure 1). Instantly I wondered what would happen if instead of squares on the sides, one constructed similar rectangles or rhombi on the sides. It was however not until fairly recently that I had an opportunity to investigate these questions with the aid of dynamic geometry.

After some initial experimentation with the arrangement of the similar rectangles and rhombi on the sides, the following two generalizations of Van Aubel were discovered using *Cabri*:

1. If similar *rectangles* are constructed on the sides of any quadrilateral as shown in Figure 2, then the centers of these rectangles form a quadrilateral with *perpendicular* diagonals
2. If similar *rhombi* are constructed on the sides of any quadrilateral as shown in Figure 3, then the centers of these rhombi form a quadrilateral with *equal* diagonals.

In both cases, it was very easy to click and drag any of the vertices of *ABCD* around the screen to see if *EG* remains perpendicular to *HF* in the first case, and in the second case whether they always remain equal. In fact, I also used the property checker of *Cabri* to verify that both results were indeed true, e.g.: "*this property is true in a general position*".

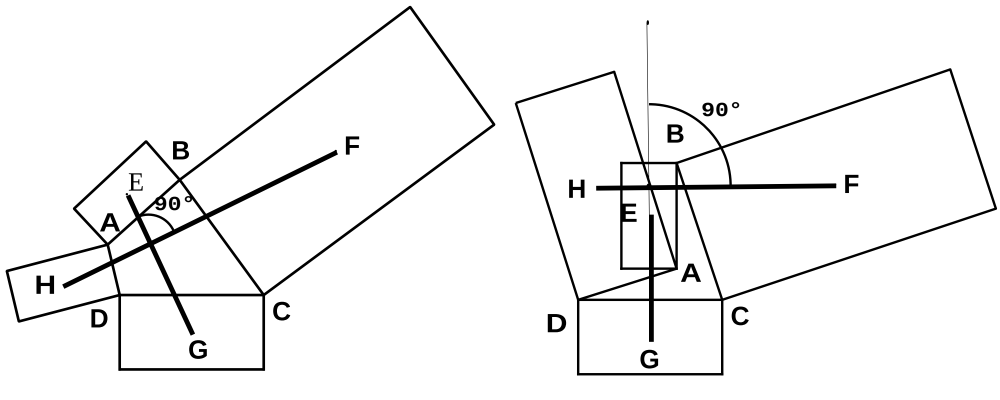

Figure 2

Armed with conviction that the generalizations were indeed true, I then proceeded with the task of constructing deductive proofs. Why did I still feel a need to prove the above results if I was already convinced of their truth?

Firstly, it is important to point out that it is precisely because I was convinced of their truth that I felt challenged to find deductive proofs, not because I doubted the results. Why? Well, here were two results that were obviously true and I was intrigued to try and find out *why* they were true. I therefore experienced the search for and eventual construction of deductive proofs as an intellectual challenge, and satisfying a deeper need for understanding, definitely not as an epistemological exercise in trying to establish their respective "*truths*". In other words, I did not really experience a need for further certainty, but rather of *explanation* (why were they true?) and of *intellectual challenge* (can I prove them?).

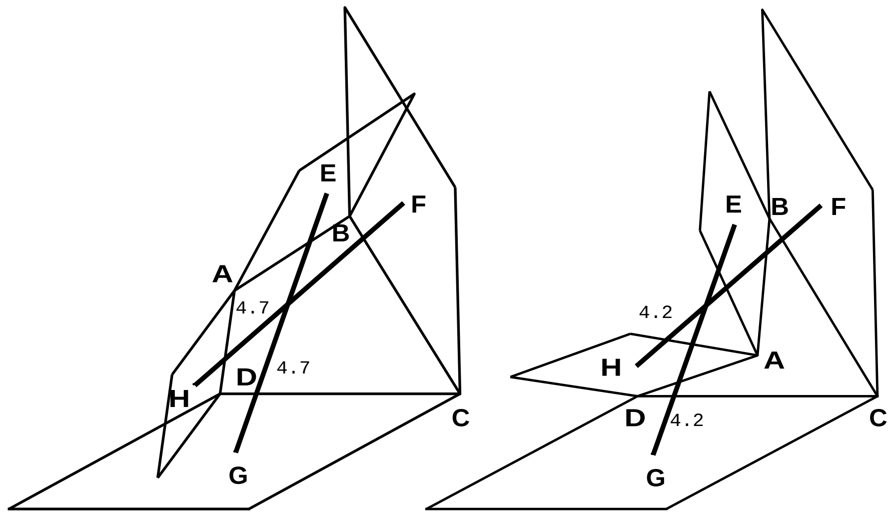

Figure 3

Doug Hofstadter (1997: 10) similarly emphasizes how conviction within a dynamic geometry context can precede and motivate proof:

> "*By the way, note that I just referred to my screen-based observation as a "fact" and a "theorem". Now any redblooded mathematician would scream bloody murder at me for referring to a "fact" or "theorem" that I had not proved. But that is not my attitude at all, and never has been. To me, this result was so clearly true that I didn't have the slightest doubt about it. I didn't need proof. If this sounds arrogant, let me explain. The beauty of Geometer's Sketchpad is that it allows you to discover instantly whether a conjecture is right or wrong - if it's wrong, it will be immediately obvious when you play around with a construction dynamically on the screen. If it's right, things will stay "in synch" right on the button no matter how you play with the figure. The degree of certainty and confidence that this gives is downright amazing. It's not a proof, of course, but in some sense, I would argue, this kind of direct contact with the phenomenon is even more convincing than a proof, because you really see it all happening right before your eyes. None of this means that*

> *I did not want a proof. In the end, proofs are critical ingredients of mathematical knowledge, and I like them as much as anyone else does. I am just not one who believes that certainty can come **only** from proofs.*"

In situations like the above, the function of a proof for the mathematician clearly cannot be that of verification/conviction, but has to be looked for in terms of other functions of proof such as explanation, intellectual challenge, etc. Although it is possible to achieve quite a high level of confidence in the validity of a conjecture by means of empirical verification by hand or computer (for example, accurate constructions and measurement, numerical substitution, and so on), this generally provides no satisfactory explanation why the conjecture may be true. It merely confirms that it is true, and even though considering more and more examples may increase one's confidence even more, it gives no psychological satisfactory sense of illumination - no insight or understanding into how the conjecture is the consequence of other familiar results.

Similarly, Davis & Hersh (1983: 369-369) present "heuristic evidence" in support of the still unproved Riemann Hypothesis, and conclude that this evidence is "*so strong that it carries conviction even without rigorous proof.*" They nevertheless express a need for proof as "*a way of understanding **why** the Riemann conjecture is true, which is something more than just knowing from convincing heuristic reasoning that it **is** true.*"

Interestingly as reported in Vimolan & De Villiers (2000), young children also displayed a need for an explanation (deeper understanding) for a result, independent of their need for conviction, which had been fully satisfied by exploration on *Sketchpad.*

Proof as a means of discovery

It is often said by critics of the traditional deductivist approach in teaching geometry that theorems are mostly first discovered by means of intuition and/or empirical methods, before they are verified by the production of proofs. However, there are numerous examples in the history of mathematics where new results were discovered or invented in a purely deductive manner; in fact, it is completely unlikely that some results (for example, the non-Euclidean geometries) could ever have been chanced upon merely by intuition and/or only using empirical methods. Even within the context of such formal deductive processes as axiomatization and defining, proof can frequently lead to new results.

To the working mathematician proof is therefore not merely a means of verifying an already-discovered result, but often also a means of exploring, analyzing, discovering and inventing new results. Indeed quite frequently explaining (proving) why a result is true enables further generalisation as shown by the following example.

In Honsberger (1985, 32-33) the reader is introduced to the so-called "*equilic quadrilateral*", namely a quadrilateral *ABCD* with one pair of opposite sides equal, say *AD* = *BC*, which are inclined at 60˚ to each other. (The latter condition might also be stated in the form $\angle A + \angle B = 120°$). Then one of the engaging results which is proved is the following: "If *ABCD* is an equilic quadrilateral and equilateral triangles are drawn on *AC*, *DC* and *DB*, away from *AB*, then the three new vertices, *P*, *Q* and *R* are collinear" (see Figure 4).

As before, I again wondered what would happen if *ABCD* was any quadrilateral with opposite sides equal and triangles *PAC*, *QDC* and *RDB* similar to each other. Would *P*, *Q* and *R* then still be collinear?

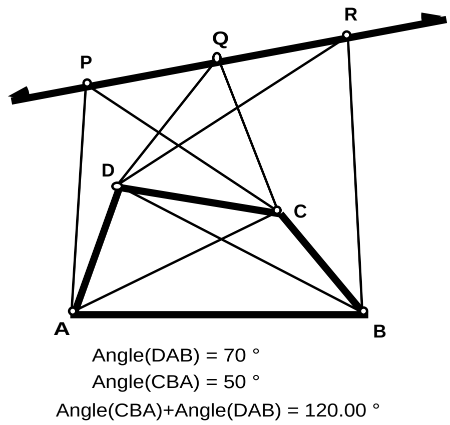

Figure 4

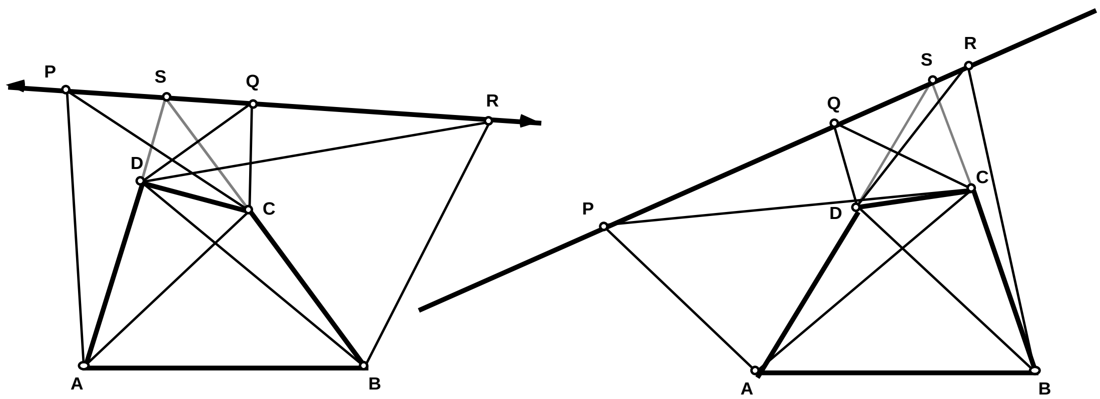

Figure 5

By investigating these questions with *Sketchpad*, I managed to discover the following interesting generalization:

"If similar triangles *PAC*, *QDC* and *RDB* are constructed on *AC*, *DC* and *DB* of any quadrilateral *ABCD* with *AD* = *BC* so that $\angle APC = \angle ASB$, where *S* is the intersection of *AD* and *BC* extended, then *P*, *Q* and *R* are collinear" (see Figure 5). It should be noted that the condition that $\angle APC = \angle ASB$ may also be alternatively stated as $\angle PAC + \angle PCA = \angle A + \angle B$ or $\angle APC = 180° - \angle A - \angle B$.

Furthermore, it was found that the point *S* is also collinear with the other three points. Using a dynamic construction with *Sketchpad* as shown in Figure 5, and varying either angle *A* or *B*, or the shape of the similar triangles, it was easy to see that the result was true in general.

Interestingly, after one or two unsuccessful attempts at proving this result, I then noticed while manipulating the configuration, that $\angle ACB = \angle APQ$, thereby enabling me to construct an eventual proof. This shows how investigation by dynamic software can also sometimes assist in the eventual construction of a proof.

However, carefully looking back at my final proof *a la Polya*, I suddenly realized that I had never used the property that *AD* = *BC*! In other words, the result was immediately generalizable to *ANY* quadrilateral! This illustrates the value of an explanatory proof which enables one to generalize a result by the identification of the fundamental properties upon which it depends. It seems unlikely that I would've found the general case purely by empirical investigation.

Proof as a means of verification (justification)

Of course, in view of the well-known limitations of intuitive, inductive or empirical methods themselves, the above arguments are definitely not meant to disregard the importance of proof as an indispensable means of verification, especially in the case of surprising, non-intuitive or doubtful results.

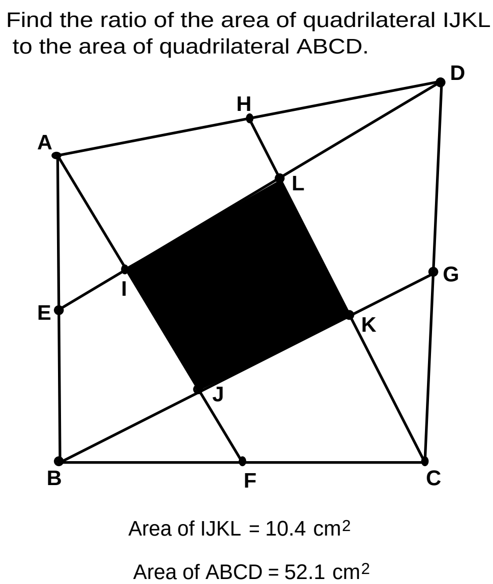

Figure 6

Therefore, even though I don't believe the verification function should be the starting point for introducing novices for the first time to proof in a dynamic geometry context, I do believe that it can be (and should be) developed later to enable students achieve a more mature understanding of the value and nature of deductive proof. Consider the following example from De Villiers (1999) which experience has shown works well with students to get this point across.

Students are given the *Sketchpad* sketch in Figure 6 to determine the ratio of the areas, to investigate further and make a conjecture. As a deliberate pedagogical device the measurement accuracy had been set to only one decimal accuracy. So no matter how much the students drag the quadrilateral, the ratio appears to be constant at 0.2.

In fact most students are quick to say that they are 100% certain. They are therefore quite taken aback when asked to increase the measurement accuracy, and then find to their great surprise that the 2nd and 3rd decimals are changing, but due to rounding off to 1 decimal this is not apparent.

This example therefore works well to sensitize students to the fact that although *Sketchpad* is very accurate and extremely useful for exploring the validity of conjectures, one could still make false conjectures with it if one is not very careful. Generally, even if one is measuring and calculating to 3 decimal accuracy, which is the maximum capacity of *Sketchpad 3*, one cannot have absolute certainty that there are no changes to the fourth, fifth or sixth decimals (or the 100th decimal!) that are just not displayed when rounding off to three decimals. This is why a logical explanation/proof, even in such a convincing environment as *Sketchpad*, is necessary for absolute certainty.

Proof as a means of communication

Several authors have stressed the importance of the communicative function of proof, for example:

> "*... we recognize that mathematical argument is addressed to a human audience, which possesses a background knowledge enabling it to understand the intentions of the speaker or author. In stating that mathematical argument is not mechanical or formal, we have also stated implicitly what it is ... namely, a* ***human interchange*** *based on shared meanings, not all of which are verbal or formulaic.*" (bold added) - Davis & Hersh (1986:73).
>
> "*... definitions are frequently proposed and argued about when counterexamples emerge ...*" - Lakatos (1976:16)

Consider for example the following activity from De Villiers (1999) in relation to the well-known theorem that the sum of the angles of a quadrilateral is always 360°.

Construct a quadrilateral *ABCD* and measure its angles. Drag vertex *D* over side *AB* to obtain a figure similar to the one shown in Figure 7. Is the sum of its interior angles still equal to 360°? Is the figure *ABCD* a "*quadrilateral*"? What do we mean by the concept "*quadrilateral*"? How does this relate to the well-known result formulated above? What do we mean by "*interior"* angles?

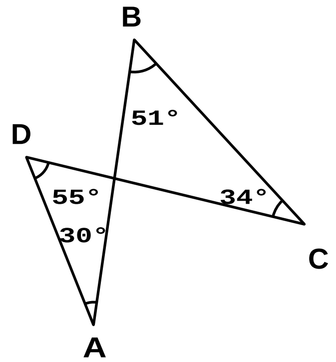

Figure 7

Most peoples' first reaction to such a "*counter-example*" is one of "*monster-barring*" in support of the theorem that the sum of the interior angles of **all** quadrilaterals is 360°, i.e. to reject figures like these as quadrilaterals. They might therefore try to define a quadrilateral in such a way that figures like these are excluded. Lakatos (1976:16) describes a similar situation after the discovery of a counter-example to the Euler-Descartes theorem for polyhedra by the characters in his book who then vehemently argue about whether to accept or reject the counter-example.

This happens because refutation by counter-example often depends on the meaning of the terms involved and consequently definitions are frequently proposed and argued about. The point is that within dynamic geometry, students are likely to accidentally construct a crossed quadrilateral by dragging and the question then arises naturally whether it is a quadrilateral or not, and what we mean by a quadrilateral. How can we define quadrilaterals precisely? What do we mean by "*interior*" angles? How can we "*save*" or "*reformulate*" the original theorem?

Proof as a means of intellectual challenge

To mathematicians proof is an intellectual challenge that they find as appealing as other people may find puzzles or other creative hobbies or endeavours. Most people have sufficient experience, if only in attempting to solve a crossword or jigzaw puzzle, to enable them to understand the exuberance with which Pythagoras and Archimedes are said to have celebrated the discovery of their proofs. Doing proofs could also be compared to the physical challenge of completing an arduous marathon or triathlon, and the satisfaction that comes afterwards. In this sense, proof serves the function *of self-*

realization and *fulfillment.*

Proof is therefore a testing ground for the intellectual stamina and ingenuity of the mathematician (compare Davis & Hersh, 1983:369). To paraphrase Mallory's famous comment on his reason for climbing Mount Everest: "*We prove our results because they're there.* Pushing this analogy even further: it is often not the existence of the mountain that is in doubt (the truth of the result), but whether (and how) one can conquer (prove) it!

Proof as a means of systematisation

Proof exposes the underlying logical relationships between statements in ways no amount of empirical testing nor pure intuition can. Proof is therefore an indispensable tool for systematizing various known results into a deductive system. Rather than providing students with ready-made definitions, I am of the opinion that they should engage in defining some mathematical concepts themselves. Already in 1908 Benchara Blandford wrote (quoted in Griffiths & Howson, 1974: 216-217):

> "*To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational.*"

Suppose for example we want to formally define the concept of rhombus, then we might proceed by first evaluating the following possibilities by construction and measurement on *Sketchpad* (compare Govender & De Villiers, 2002):

(a) A rhombus is any quadrilateral with perpendicular diagonals.

(b) A rhombus is any quadrilateral with perpendicular, bisecting diagonals.

(c)A rhombus is any quadrilateral with two pairs of adjacent sides equal.

Such an investigation easily shows that the first and last ones above are deficient, but no matter how we drag the constructed rhombus in the second case, it always remains a rhombus. This implies that the conditions contained in (b) are sufficient, and that one should be able to accept this statement as a formal definition, and logically derive (prove) all the other properties of a rhombus as theorems (e.g. all sides are equal, etc.)

Concluding comments

Rather than one-sidedly trying to focus on proof only as a means of verification in dynamic geometry (which does not make sense to novice students), the more fundamental function of explanation and discovery ought to be initially utilized to introduce proof as a meaningful activity to students. This requires that students should be inducted early into the art of problem posing and allowed sufficient opportunity for exploration, conjecturing, refuting, reformulating, explaining, etc. Dynamic geometry software strongly encourages this kind of thinking as they are not only powerful means of verifying true conjectures, but also extremely valuable in constructing counter-examples for false conjectures.

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