

Students' Complex and Nested Uses of Shared Representations: An Example with Linear Functions in a Technological Environment

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Abstract

We used interviews to elicit individual students' deep understandings and ways of working with and on shared representations¹ of mathematical things. Eleven high school students were interviewed on three occasions, once in each of February, March, and April. The interviews done in the context of this study exposed the learners' intricate work with shared representations, technology, and mathematical content. We report here the extent to which the MAGICAL framework was useful in analyzing students' actions on and with representations of functions and related mathematical objects. We organize the discussion of MAGICAL activity in light of the tasks we offered and students pursued. Embedded in this structure are details of the students' purposes in using various types of technology. Within the discussion of the representation acts, we find nested patterns of representation work that illuminate the complexity of student thinking with and about shared representations.

Research Questions

One overarching question of this set of papers is *What is the nature of students' use of representations when they solve mathematical problems with access to mathematical tools?* In the interviews we wanted to see how students reacted to representations we shared with them as well as what representations they would share with us. We use the interview data to answer the question in terms of the patterns of students' representation work. To show the integrated nature of the students' interactions with different representations and different aspects of a representation, we display these patterns in diagrams that show a nesting effect.

Interview Tasks and Their Influence

As experienced interviewers, we suspected that the *interview tasks* – the tasks we posed to the students – would frame the students' activity. Paralleling Doyle's (1988) comments on tasks in the classroom, we note a task in the interview could exist concurrently in several different forms. These forms include what the interviewer poses as the task, what the student interprets the task to be, and what the interviewer interprets as the task based on the student's response. The task may also vary over time as the student receives feedback from the technology as well as probes from the interviewer.

¹ Shared representations “are potential representations of mathematical ideas and they are available to students and to others in the setting” (Zbiek, 2002a, p. 2). They are external representations but they are not necessarily the external (as opposed to internal) representations of any one individual.

But we agree with Dugdale, Thompson, Harvey, Demana, Waits, Kieran, McConnell, and Christmas (1995) as they counted focusing student attention “by suggesting questions, possible avenues of inquiry, and work strategies” as a quality of an activity (p. 343). The posed interview tasks may differ at times from the tasks the student undertakes. We find sufficient influence to justify using the posed tasks as organizers for looking at interview excerpts.

As noted by Heid, Blume, Hollenbrands and Piez (2002), tasks posed in the CAS-Intensive Mathematics setting may be of several types. Using the categories posed by Heid and her colleagues, some tasks are concept-related tasks (Identify, Describe, and Compare/Elaborate/Describe Phenomenon). Other tasks have students produce something ranging from products of well-known algorithms, to more creative solutions, through conjectures and generalizations (i.e., Produce, Generate, Predict, and Generalize). There are also reasoning tasks (Corroborate and Justify) that follow the production or introduction of a mathematical statement. We used these task types to describe the main interview tasks we gave students. To illustrate our points in this paper, we draw on the February interview.

The principle items in the February interview appear in Figure 1. The details of the representations involved in these items appear in the next section of this paper. We considered the relationship between the task types and the MAGICAL categories. The Compare/Elaborate/Describe (CED) tasks elicited Linking, Connecting, Ascribing, Generating, and Interpreting actions. Describe Observation (DO) tasks led to Connection and Interpreting actions. Generate Function Specifics (GFS) and Produce Graph (PG) items corresponded to Generate and Link actions. These combinations of types of tasks and corresponding representation actions were common across the interviews. The bulk of the student-researcher talk in the interviews clearly centered on CED items. To elaborate further on the complex nature of students’ work with shared representations as elicited in our interviews, we will focus on two examples of CED items and the representational use with technology that they elicited. The first example involves Interpretation acts arising from a CED item in the similarities and differences style of Item 1) in Figure 1. The second example centers on Linking acts arising from a CED item of the matching genre of Item 5) in Figure 1.

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- 1) [*Compare/Elaborate/Describe*] Does this [a linear or quadratic dynamap] look similar to anything that you have seen or talked about it in class?
 - 2) [*Describe Observations*] When you move this one [the input point of the dynamap], how would you describe what happens?
 - 3) [*Generate Function Specifics*] What do you think the rule for this function would look like?
 - 4) [*Justification*] How did you know what to write [for the rule]?
 - 5) [*Compare/Elaborate/Describe*] Could any of these expressions [reproduced to the right] be used for the rule of the function shown on the screen [in the linear dynamap]?
 - 6) [*Produce Graph*] What would the usual [probably Cartesian] graph of this function look like?

$$\begin{array}{c}
 8K - 4 \\
 -2K + 6 \\
 K^2 - 5 \\
 -K^2 + 7 \\
 K + 3 \\
 9
 \end{array}$$

Figure 1. Task types of the principle tasks in the February interview schedule

Representation Potential in the Interviews

The interview tasks involved several different types of shared representations as posed and provided likely opportunities for several other types of shared representations. The principal interview tasks for the February session (See Figure 1) involved dynamaps. Our dynamaps were a variation on DynaGraphs (Goldenberg, Lewis, & O’Keefe, 1992). For example, Figure 2 shows the initial (static) image of the linear dynamap presented in Item 1). Students drag point K and see the position of point g(K) change accordingly. In this way each static image represents only one ordered pair belonging to the function. As a representation of a function, the dynamap is what we call a *representation over time*. The function relationship lies within the series of these static images that students experience over time. By adding the trace of the arrows, as illustrated for the linear function dynamap in Figure 3, students have an alternative to the Cartesian graph as a global representation of a function.

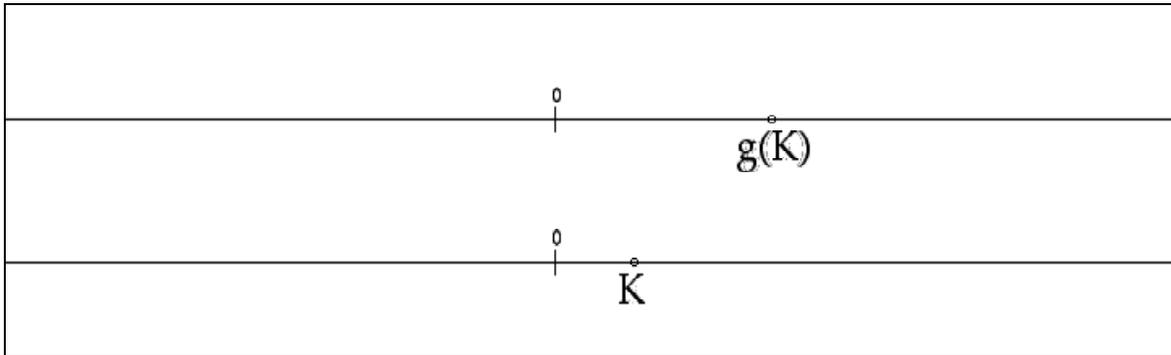


Figure 2. Initial static image of linear dynamap from February interview

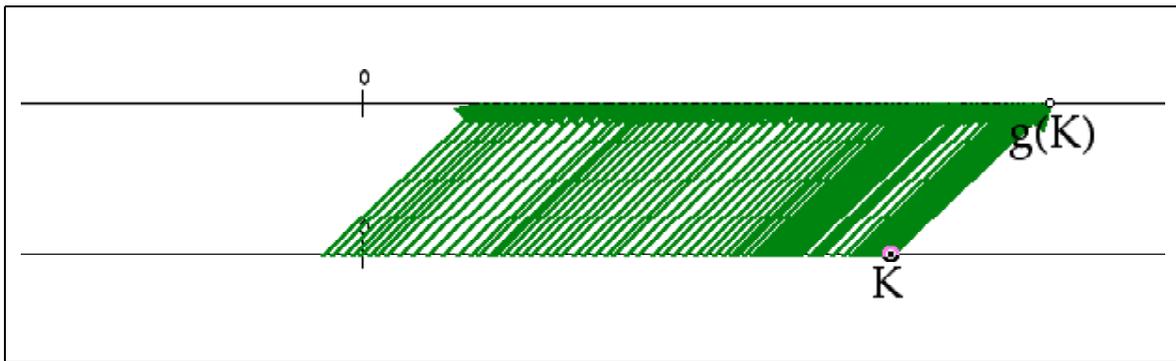


Figure 3. Trace sample for g (the particular linear function given by $g(K)=K+3$)

It seems appropriate to note also the representations in the tasks posed in the subsequent Spring 2001 interviews. The March interviews involved dynamaps. However, the basic dynamaps used in the February sessions were extended to include two dynamaps on the same screen (as in Figure 4) in explorations of inverse functions and composition of functions. Tasks in the April interview used a slidergraph, our name for a dynamic representation for a family of functions that we based on Cartesian graph (see Figure 5). “Sliders” at the bottom of the screen allow the user to control the values of the family parameters to obtain a particular member of the function family. In addition to displaying the Cartesian graph of the family member, the slidergraph screen shows the general symbolic form of the function family and the symbolic rule for that particular function family member. The parameter values in the specific member rule are updated automatically as the user drags the sliders.

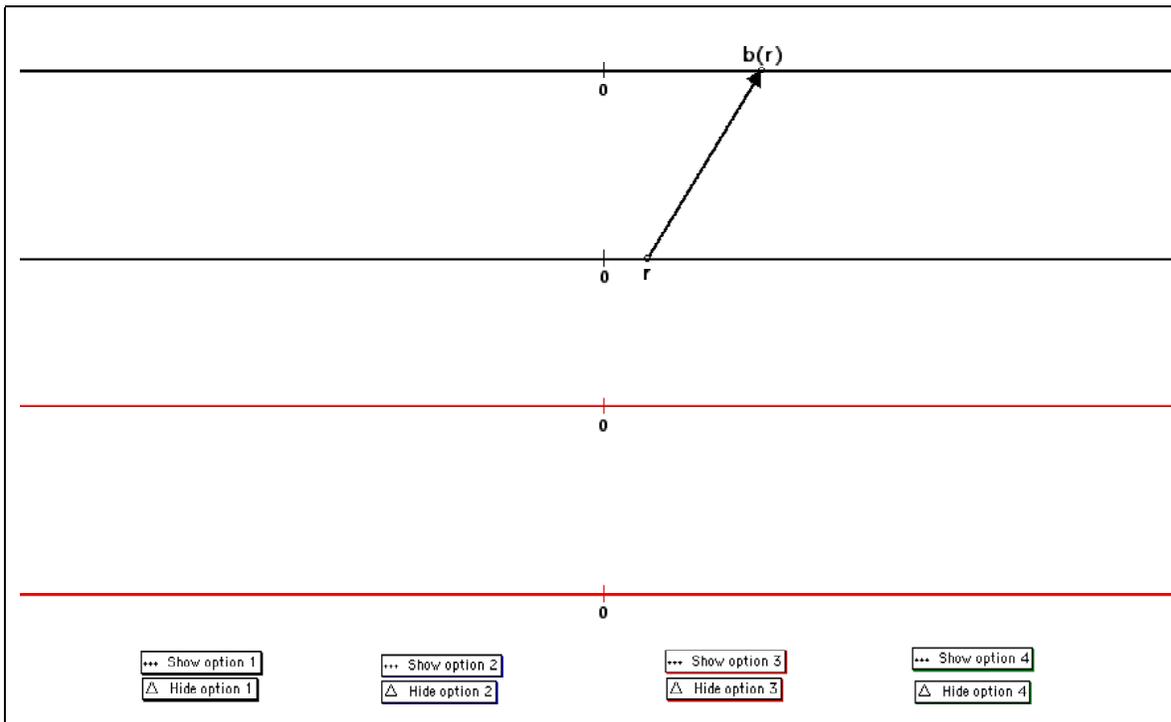


Figure 4. Sample of paired dynamaps to represent potential inverse functions

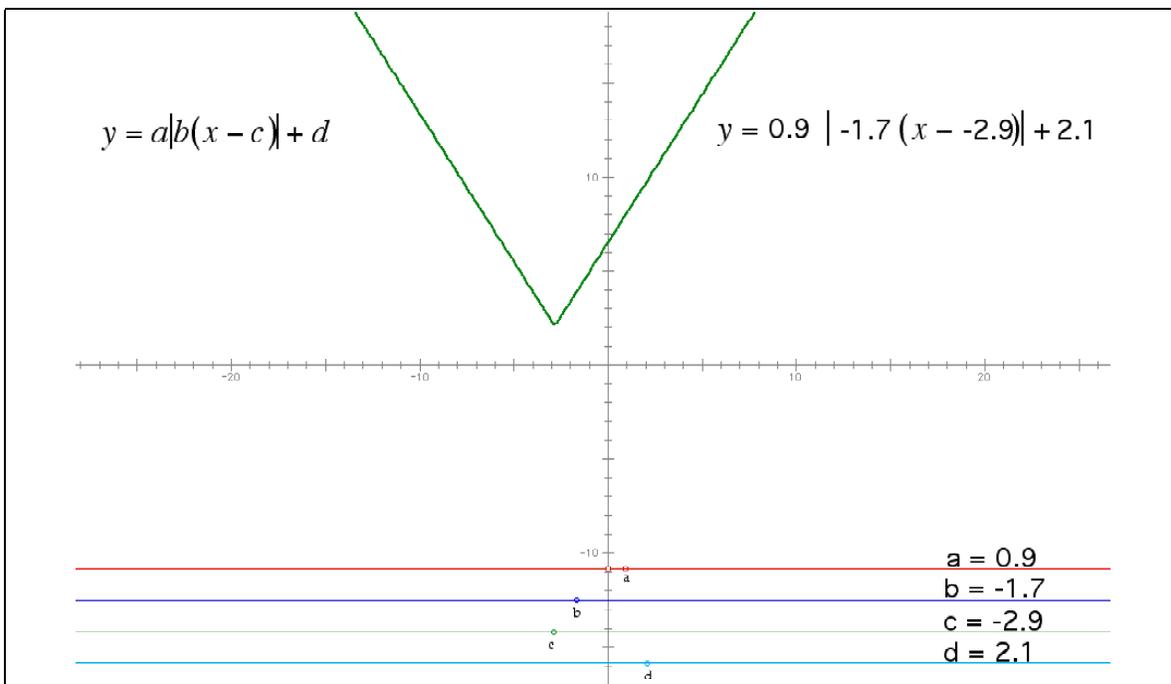


Figure 5. Static view of slidergraph for absolute value function family from the April interview

All three interviews afforded opportunities for students to view and to create (via multiple forms of technology) other representations of function and of other concepts. Among those representations used frequently were Cartesian graphs, symbolic expressions, and physical motions. In addition to representing specific functions as well as prototypical members of a function family, students represented other mathematical things. The most common other things were geometric objects (e.g., lines of reflection) and mathematical relationships (e.g., rates of change).

We chose to present these tasks through *The Geometer's Sketchpad* (Jackiw, 1991). As Goldenberg (1995) observed, this particular tool choice likely skewed the picture of the students' internal representations. This is why we prefer to think about looking at the students' work with shared representations. As learners, the students are developing understandings. Their work with shared representations includes a blend of what is possible given their current understandings as well as what they choose to discuss and to share representationally. We note that the representations students share can be reflective of their understandings in progress. As a result, the nature of their interpretations of their shared representations may be unstable across or within a given task.

Technology Potential in Interview Setting

The interview setting offered potential for technology use in several ways. Some of the tasks were presented via electronic files with which students could interact. The interviewee had unrestricted access to paper and pencil, a symbolic calculator (TI-89), and a computer with *The Geometer's Sketchpad*. The interviewer also had access to this same set of tools but worked under a general principle of not introducing technology use other than opening files that conveyed tasks.

We coded students' use of these tools as well as their use of body motions and physical objects (e.g., hands being used to create a V-shape) for the purposes that students or interviewers had in using the technology. These purposes were identified using the five main categories (checking answers, getting information, delegating work, getting solutions, and improving presentation) and 16 subcategories as outlined by Zbiek (2002b).

Jim as a Student and as an Interviewee

We will use two transcript excerpts from the February interview with Jim (a pseudonym) to illustrate the complexity and the MAGICAL nature of students' use of shared representations. We chose Jim's work for several reasons. He was one of the students who completed all three interviews with ample time to present his understandings. He was a consistent participant in one small group during all of the

small-group observations. The examples from Jim's interview are also sufficiently concise for the constraints of this paper.

Jim was a Grade 10 male who scored at the 66 percentile in mathematics and at the 63 percentile in reading on the Comprehensive Test of Basic Skills taken during the previous school year. He has been earning grades of B and A in high school mathematics. Although somewhat shy at the start of the interview setting, Jim was articulate and willing to discuss his ideas. His willingness to share was essential for our purpose of thinking about what interviews offer as well as for considering how the MAGICAL framework can illuminate students' use of shared representations in a mathematics technology setting.

Representation Actions as Nested Phenomena

We use an example from Jim's February interview to illustrate the complexity of student thinking about shared representations as captured through the MAGICAL framework. We also comment on the purposes that technology serves in meeting the representation goals of the student and of the interviewer. The foci are the representations Jim chose to share with the interviewer, how Jim understood and related those representations, and how Jim interpreted and used the representations shared by the interviewer.

Jim's work in linking multiple representations involved looking at particular components of a type of representation through a certain lens. He associated specific events in the dynamaps with relative positions in the Cartesian graphs. However, he related events and positions to each other by connecting each with component characters of the symbolic representations. As we discuss the transcript excerpt, these connections, the pattern of nesting and changes in nesting over time become more apparent.

Prior to this excerpt, Jim had been asked to drag point K and to describe what happened. The task in this excerpt is to match one or more of six expressions (shown in the following excerpt and in the box in Figure 1) with the dynamap (shown in Figure 2). Representationally speaking and using the representation acts discussed by Zbiek (2002a), the student responds with a Linking act (relating parts of two different types of representations) involving Different types of representations (dynamaps and symbolic expressions) of Different mathematical things (function g embodied in the dynamap and the functions underlying the expressions on the paper).

I: *[Jim had written " $g(k) =$ " on a paper and stopped.]* These are a couple of possibilities of what we might put down there after the g of K. *[I shows S sheet of expressions shown below.]* Out of all of those, which, if any of them, do you think could be possible rules for this one?

$$\begin{array}{r}
 8K - 4 \\
 -2K + 6 \\
 K^2 - 5 \\
 -K^2 + 7 \\
 K + 3 \\
 9
 \end{array}$$

S: I'd think K plus three.

Not using any technology other than pointing at objects on the screen, Jim immediately picked $K+3$. At this point he is Linking the expression and dynamap. In the absence of Jim providing an elaboration about the shared representations, the next diagram begins simply as shown in Figure 6. [We will use a three-part style in conveying parts of the diagram. The first line will be the MAGICAL code (e.g., DDL). The last line gives the relevant line numbers of the transcript. The intervening part is a description of the representation activity.]

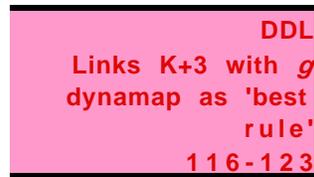


Figure 6. Diagram for *Which, if any of these expressions could be possible rules for this (g) dynamap?*

Note that the interviewer then moves from a CED task to a Justification or Corroboration task by asking Jim why his answer made sense.

I: Mmm-hmm.... Because?

S: Ah...because...K plus three means that they'll stay equal distance apart no matter where you move them. And they'll move the same distance. And if...if it was any of the other ones, it wouldn't look like that. They wouldn't – they wouldn't stay the same...on the two lines.

Jim is focused on the function given by the expression, $K+3$. His Linking in this last passage depends on the claim that this particular expression meant that, as he would drag the input point K , the distance between the output and input points would always be the same and the output point would always move the same distance as the input moved. We diagram this shared representation use as in Figure 7. Jim posited and shared the new dynamap by verbalizing its existence. [We note these students were not expected to be able to create their own dynamap files.]

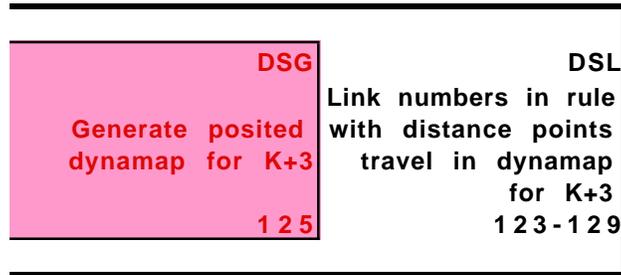


Figure 7. Jim's work with K+3 when encouraged to explain why it was the best expression for the given dynamap.

Jim seemingly is Linking the dynamap with the chosen expression by considering the distances between the input and output points in the dynamap for the function given by K+3. The interviewer suspected Jim might be (correctly) using the slope 1 for K+3 to conclude the output point moves as far as the input point. She tested this suspicion by asking Jim to talk about another linear dynamap, 8K-4.

- I: Mmm-hmm. So, for example, what would the two lines look like if it was, um, the first one here [*points to expression 8K-4*]?
- S: Oh...if it was eight K minus four –
- I: Mmm-hmm.
- S: Ah...g of – actually...ah...g of K would be... [*4 seconds*]
I don't know really. Ah...I guess g of K would be...eight...times farther away from the starting point than ... K, but then you'd have to subtract four because of the minus four.

Jim extends his Linking work with the expressions and dynamap. In particular he is focused on one expression, 8K-4. This Linking depends on seeing the "8" in the expression as the factor to multiply the distance between 0 and K in the lower dynamap line and then subtracting 4 to obtain the distance between 0 and g(K) on the top dynamap line. We diagram this representation use as shown in Figure 8.

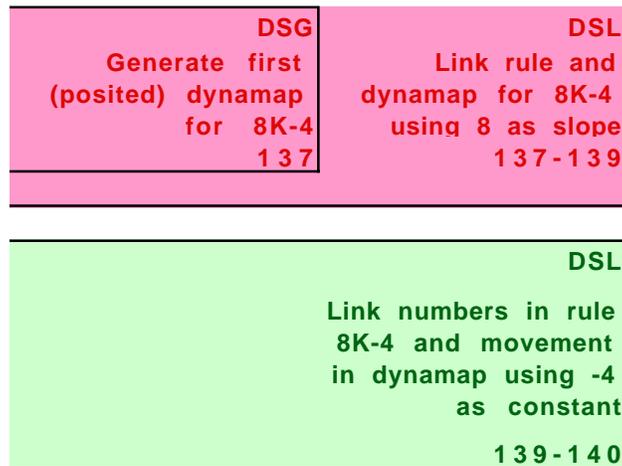


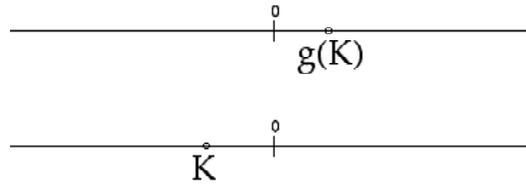
Figure 8. Response to interviewer’s query, *What would the dynamap look like for 8K-4?*

Jim’s response is consistent with seeing the horizontal distance between 0 and $g(K)$ on the output line in terms of a multiple of the horizontal distance between 0 and K on the input line adjusted for the constant, -4 . He still uses no technology other than pointing at locations on the screen. The interviewer asks Jim to discuss a particular input/output pair to test this hypothesis.

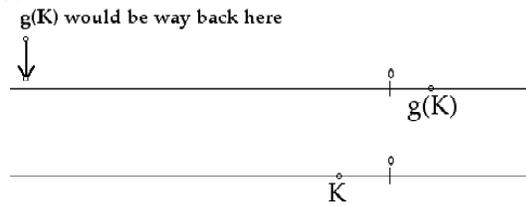
- I: Uh-huh.... So, if K was right here [points to point K in current position] –
 - S: Mmm-hmm.
 - I: Where would you predict g of K would be?
 - S: Probably...like...um, I don't know...like way out here, or something. Like further away.
-
- I: To get that eight times as far away?
 - S: Mmm-hmm.

The interviewer notes the multiple of 8 but further questions how Jim sees the sign of the value of the input. She is aware other students interviewed had visualized a line through K and $g(K)$ and used a linear coefficient like 8 in $8K-4$ as the slope of this visualized line. To probe Jim’s understanding on this aspect, the interviewer asked Jim to predict the results for a negative input value.

- I: Okay. And then if we move K – let's say we move K over here [points to a location on the input line left of 0]. Where would you expect g of K to be then?
- S: Well then if you move there then...[S moves point K to suggested position]



You get eight times negative K. So, then it would be way back here because...a negative times a positive is a negative. So that would make it way back there.



Jim's Linking of dynamaps and expressions resided in connections he shared between the expression as a calculation and the horizontal directed distance from 0 to $g(K)$.

Until this time in the interview, Jim used the dynamap file primarily by pointing to locations on the screen to illustrate his expectations about where points would be and how they would move. When the interviewer asked him to predict the output for an input point "over here" in the negative values, Jim dragged the input point K to represent a negative value. The given dynamap did not represent the function denoted by $8K-4$, and Jim seemingly knew that fact. It appears that Jim is creating a representation of a particular input point as part of a dynamap for the $8K-4$ function. This seems to be done in order to advance his thinking in terms of calculations. Integrating these specific examples of shared representation use in the emerging nest diagram yields Figure 9.

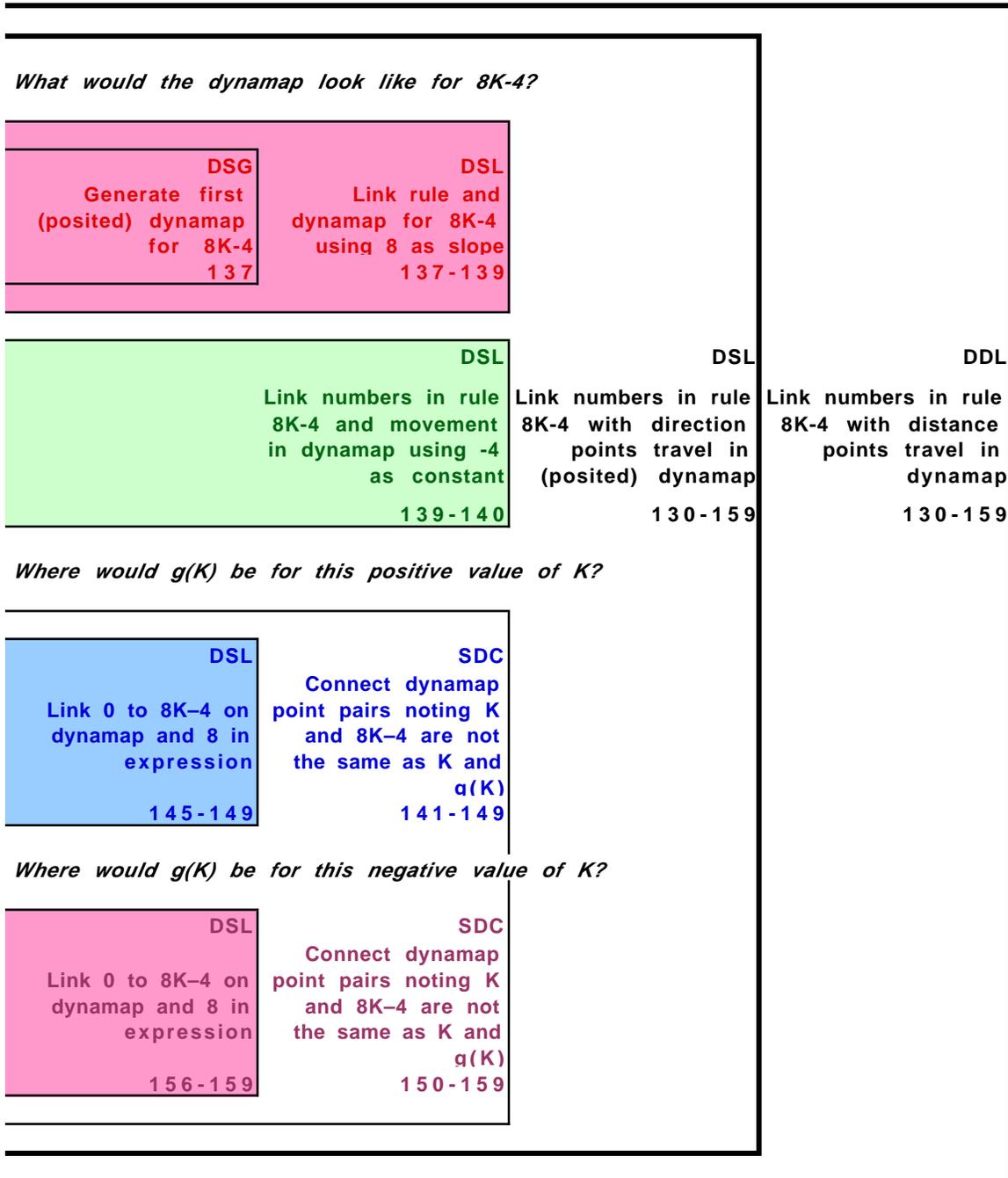


Figure 9. Adding Jim’s responses to two questions (*Where would $g(K)$ be for this positive value of K ? a negative value of K ?*) to complete the nest diagram for his discussion of 8K+4.

An essential feature of this nested pattern is that the student is doing mental calculations of output values and then using representations of input and output points in order to discuss the Linking of the dynamap for g and the expression 8K-4.

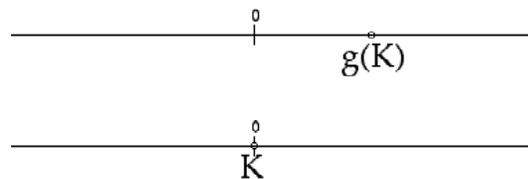
The interviewer then moved to the other expressions on the printed list in order to see the extent to which Jim's calculation work might unfold.

I: Okay.... Okay. Um, we talked about K plus three is probably the best one on there. We just talked about the eight K minus four. What about these other ones? Are there ways in which you could rule those out?

S: Ah...the negative two K plus six...ah, that might work, but I don't...because...if...negative two...that would mean that g of K would be back two times as much as K . But...then you add six. So, it really depends on what number K would be on.

I: Mmm-hmm.

S: Because...if it was, like, on ten...then, ah...If it was on ten then, it'd be like...uh...be negative twenty plus six which would be...negative fourteen. But if it's on, like, two, then it would be a different number. So, that wouldn't cause them to move evenly like this. [*S moves point K to 0*]



Because this one [$g(k)$] would move a lot – a greater distance than K would.

I: For that negative two K plus six?

S: Yeah.

Jim again uses the electronic technology to drag point K as a way to represent a particular input point and then posits the location of the corresponding output point. He also does mental calculations. By creating input/output point pairs over time, he in a sense is generating the posited dynamap of the function represented by $-2K+6$. His linking of the given dynamap for g and expression $-2K+6$ is based on treating the expression as computation linked to input and output points and their relative positions in the posited dynamap. This passage illustrates a point-wise understanding of dynamaps similar to the local rather than global understandings of Cartesian graphs noted by (Bell, & Janvier, 1981). Although the dynamap is a representation over time (Zbiek, 2002a), Jim seems to work only with snapshots consisting of particular points. He eventually does invoke the time factor as he starts to talk about moving the points. However, he seems to be looking at the dynamap as a series of discrete snapshots gathered over time. Integrating these specific examples in the emerging nest diagram yields Figure 10.

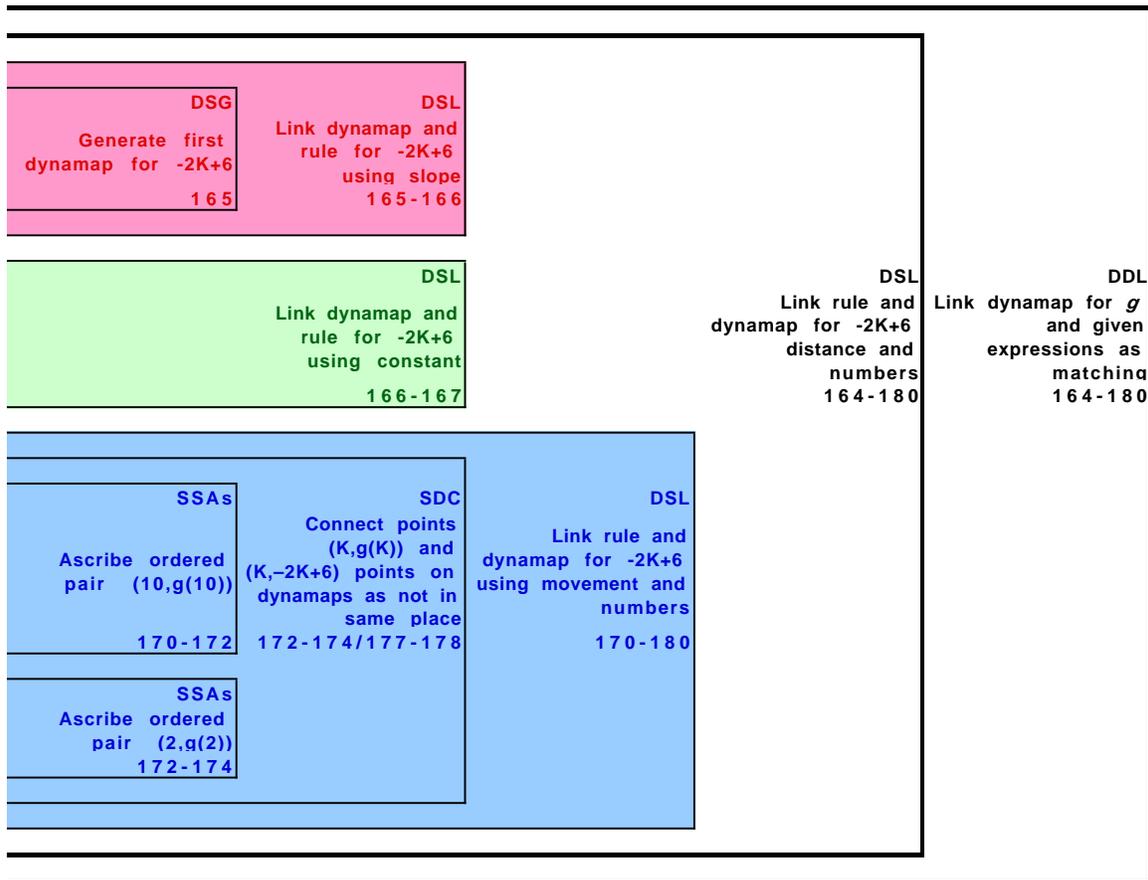
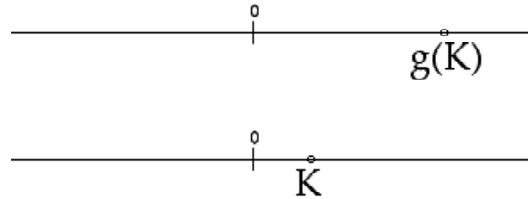


Figure 10. Jim’s discussion of $-2K+6$ in response to interviewer’s question, *Are there ways to rule out the other expressions?*

For linear expressions with different slopes, the discrete snapshots should capture the effects of the different multiples of K . To probe his understanding of this phenomenon, the interviewer asked Jim to make the comparison between the most recently considered expression $(-2K+6)$ and the third of the three expressions in the discussion thus far, $(8K-4)$.

- I: Would they [*the points for $-2K+6$*] move the same distance for the $K - \text{eight } K \text{ minus four}$?
- S: Ah...No, because...they would be equal on zero. Ah, no they wouldn’t be equal at zero. But, they wouldn’t move the same distance because when you add the multiplication in there it causes the g of K to move more. Like, it will start out further away but, like, the further you move it this way...[*S moves point K right*]



...it [point $g(K)$] will move that way but even further.

I: Uh-huh.

S: So, like the closer to zero you get the closer they will get to each other. But the further away you move, the further away they'll get from each other.

Jim temporarily tried comparing the two functions by Connecting the given dynamap and the posited dynamap in terms of the location of the input and output points for input value 0. This required Generating via mental mathematics the output points for input 0 for the generic case of the linear functions given by $-2K+6$ and $8K-4$ as shown here:

DSL
 Link generic rule
 and dynamap for a
 linear function
 using movement and
 distance from 0
 181-195

Figure 11. Jim responds generically when asked, *Would the dynamap points for $-2K+6$ move the same distance for the dynamap for $8K-4$?*

Jim continued to work with dynamaps in terms of the relative distances (and, when needed, the directed distances) output points traveled. He seemingly intermingled distance traveled with his related attention to the relative horizontal distances between the 0 on the output line and the output point. His calculations were done via mental mathematics but he used the *Sketchpad* file to point to the locations he identified.

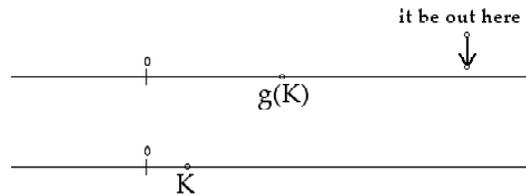
Given his consistent interpretation of the three linear expressions, the interviewer inquires about the three remaining nonlinear expressions.

I: Okay. What about the other three on there?

S: Uh. [3 seconds]

K squared minus five...would make...g of K be...[15 seconds]

It would make it be...like...it'd be way further – it'd be out here too.



Because, uh...[7 seconds]

It would – it would – it just would be further out. It'd be further out but it...it would move a greater distance...but...[10 seconds] [Note: Jim moves from talking about K^2-5 at the start of this passage to K^2+7 in the next lines; the list of expressions included $-K^2+7$ but not $+K^2+7$.]

See, it'd be...it'd be twice as far as K because it'd be K times K . So...it really depends on what K equals because it's – it'd be – if it's two, it's two times two, and that's four, plus seven is eleven. But if it's five, then it's twenty-five plus seven, which is thirty-two. So, this [point $g(K)$] would move a lot greater distance depending on what this one [point K] was.

I: Got it.

Jim's discussion of the quadratic expression was not as fluent as his talk about linear expressions but both discussions relied on using mental calculations to ascribe sample input points and interpreting the expression as a calculation. He linked the rule and dynamap for K^2-5 for relative distance in the $K+3$ dynamap. We see the same general pattern with consideration of the input/output points nested within consideration of the dynamaps in Figure 12 as we saw in Figure 10.

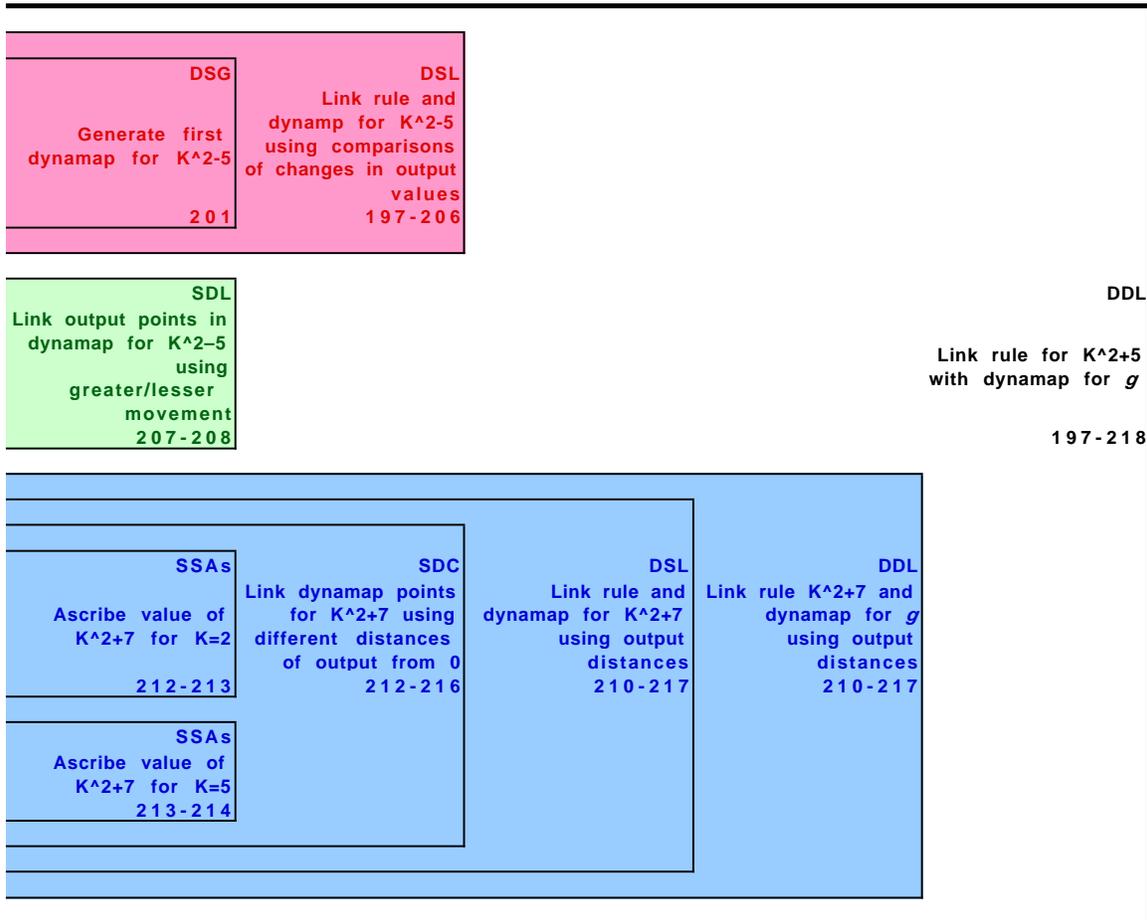


Figure 12. Discussion of quadratic expressions in response to, *What about ruling out the other three (nonlinear) expressions?*

Jim's seemingly solid explanation of the specific and generic numerical examples implies a strong understanding of whole number operations.

Quickly shifting his attention to the last expression, “9”, Jim rejected the possibility of it being a function – and therefore did not consider its dynamap – based on the absence of a variable (K).

- S: And nine.
- I: Mmm-hmm.
- S: I don't think it's possible for g of K to just equal nine.
- I: Why's that?
- S: Because you have to have K , because it's the function of K and you have to have the K in there for it...to work.

Jim's discussion of the constant-valued expression further substantiates the hypothesis that his linking of dynamaps and expressions is rooted in expressions as computations

and input-output pairs as the dominating components of dynamaps. The nesting phenomenon observed prior to this time ceases abruptly with this interpretation, as diagramed in Figure 13.



SSI
Interprets variable-
free expression 9
as not a function
220-223

Figure 13. Jim’s spontaneous comments on the constant expression, “9”

Although he could work with the interviewer’s arbitrary values of K for $8K-4$, Jim himself used specific values of K . His values of K suggested some strategic choices (i.e., use 0 as common output to compare dynamaps of two different functions, use one negative number and one positive number to check the direction from K to $g(K)$). However, his computational view of the expressions seemed to impede effective consideration of the constant function.

Vitality of the Framework

The MAGICAL perspective underscored the relationship in this example between Jim’s understanding of ordered pairs, his view of expressions as computations, and his perception of dynamaps. Interestingly, Jim chose 0 as the common input when comparing the dynamaps of two different expressions. This occurred after he considered a negative value (-2) and then two positive values (10 and 2) for input when discussing the dynamap of the single function (that given by $-2K+6$) earlier in the interview. It could be that Jim’s choice of negative and positive values was influenced by the interviewer’s request for him to consider her arbitrarily chosen positive and negative values when discussing $8K-4$ early in this exchange. However, when Jim later discussed the dynamap of the quadratic expressions, he returned to using virtually arbitrary positive values (namely, 2 and 5). Jim’s move to considering input 0 in Connecting two dynamaps rather than positive and negative numbers as input when Interpreting one dynamap may be a fundamental difference in the nest diagrams of his representation use.

The MAGICAL framework and the resulting nest diagrams particularly helped us to see similarities and differences in students’ reasoning when looking at various families of functions or expressions. For example, Figure 9 and Figure 10 show a common pattern for linear situations. These linear cases (as well as the quadratic cases) are strikingly different than the constant case shown in Figure 13.

Conclusion

The intended and posed interview tasks provided the context for the students' work. Due to the substantial extent to which these tasks framed student work, we observed the nature of representation actions within the particular types of tasks we as researchers proposed. Attending to the nested nature of the MAGICAL actions illuminated the complexity of students' actions on and with shared representations. The application of the framework also underscored students' understandings across related mathematical entities. This is exemplified by the vivid distinctions between families of functions found in Jim's nest diagrams. Some students used the technology very rarely when asked questions that led to interpretations. It seems that using the technology as a Representation Generator, as Jim often did in the example, seemed to be more frequent for students exhibiting in these interviews richer understandings of fundamental mathematical ideas.

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