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ALGEBRAIC PROCEDURES USED BY 13-TO-15-YEAR-OLDS

ABSTRACT. Students of Grade 7 were given a test followed by individual interviews; at the end of Grade 8 the same students were subject to an analogous test and interviews. Each student had to simplify certain algebraic expressions. This article is focussed on what types of procedures were used by the students performing the task and how they were explained during the interviews. The author identifies seven types of procedures used by students, labelled: (A) *Automatization*, (F) *Formulas*, (GS) *Guessing-Substituting*, (PM) *Preparatory Modification* of the expression (this includes a subtype: *Atomization*), (C) *Concretization*, (R) *Rules*, (QR) *Quasi-rules*. Part of the students' procedures led to correct results, others were wrong. Most of the procedures appeared spontaneous in the sense that they had not been taught in the classroom. Prior to the tests, the teachers (in accordance with the curriculum) had done their best to explain the validity of algebraic transformations by referring to the commutativity of addition and multiplication, distributivity, and to geometric interpretation; however, the interviewees (even explicitly asked) seldom used such arguments.

INTRODUCTION

This paper is a report on the procedures used by students (aged 13 to 15) who were to simplify given algebraic expressions or, put differently, to perform given algebraic operations, e.g.,

$$6x + 3x = \ldots \text{ or } (-2x) \cdot 8x = \ldots$$

The main questions considered here are: *What procedures are used by the students while performing such tasks? Are they mostly procedures that were explained and used earlier by the teacher? Or perhaps most procedures are spontaneous, that is, invented by children themselves? What can be said about the development of students' procedures? Are they stable over a long period? Is there a correlation between the types of procedures and the achievements of the students using them? How do the students interpret the equality sign in an algebraic identity such as* $6x+3x = 9x$*?*

Clarifying the terms

In the sequel, the term *operation* will be confined to four basic arithmetic/algebraic operations, supplemented with squaring and cubing. *Basic*

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properties of operations will mean the standard properties: the commutativity and associativity of addition and multiplication, identity properties of 0 and 1, distributivity, and relations between inverse operations (expressed as a formula, e.g., $a + b = b + a$, or verbally referring to actions).

By the structure of an algebraic expression it is meant its *surface structure* in the sense of C. Kieran (1989), i.e., the arrangement of the terms and operations. The phrase: *transforming an algebraic expression* $f(x)$ means: finding another expression $g(x)$ such that $f(x) = g(x)$ for all x in the intersection of the domains of f and g; generally, the purpose of the transformation is to find a simpler form of the expression.

According to Webster's New World Dictionary, procedure is "the act, method, or manner of proceeding in some process or course of action". In this article by a *type of procedure* it is meant the method or manner of transforming a given algebraic expression while by a *student's procedure* it is meant a particular way of doing this by the interviewee, judged by his/her performance, especially by the response to the request: "*Tell me: How did you get your result?*" together with the explanation and justification (given spontaneously by the child or prompted by explicit questions).

Background

Difficulties and errors of students' transforming algebraic expressions have been of interest to researchers for several decades. In particular, in Poland in the 1950's the problem was a subject of vivid discussions in the context of formalism in mathematics education. Krygowska (1957) gave examples of certain erroneous transformations of algebraic expressions and called the phenomenon 'formalism'; later she used a stronger term *degenerate formalism*, characterized as follows (Krygowska, 1977: p. 101):

> The consequences of not taking the due care that the students' understanding of the structure of algebraic expressions be always correct and clear, are known: glaring errors in algebraic transformations, 'degenerate' formalism, which manifests itself in thoughtless, 'slapdash' manipulation of symbols, is something totally different from correct formalism, which also consists in manipulating symbols, but in accordance to strictly applied rules.

According to Ćwik (1984: p. 76), degenerate formalism consists in so strong a detachment of the rules of manipulating symbols from their meaning that, first, referring to the meaning is no longer a way of checking the correctness of the computation and, second, the student formulates his/her own wrong rules. These latter are often persistently fixed in the mind, or may be used *ad hoc*, based on various associations, but neither on the meaning of symbols nor on formal deduction.

Turnau (1990: p. 163) considers it necessary that students move back from algebra to the intuitive meaning of objects and operations. This should happen frequently enough so that such returns be always possible.

There is a long-standing question whether school algebra should be presented the way algebra was understood in the early nineteenth century (see, e.g., Eves, 1981: p. 89 and 98), that is, as *generalized arithmetic*, governed by the same familiar laws as those concerning computations on plain numbers. According to this interpretation, in algebra – instead of working with specific numbers (as we do in arithmetic) – we employ letters which represent numbers. Or perhaps algebra should be treated as a *symbolic system* based on formal rules? Thus, two aspects of transformations of algebraic expressions should be considered: (i) the equality of expressions when numbers are substituted instead of the variables, (ii) obeying the rules (such as basic properties of operations). How should these two aspects be included in learning algebra? Though the present study is not aimed at such a general problem, answering the questions raised at the beginning of the paper may shed a light on it.

An account of arithmetical roots of the early algebra learning was given by C. Kieran in her survey (1989). She pointed out that the traditional emphasis in the curriculum on 'finding the answer' allows arithmetic learners to get by with informal, intuitive procedures; however, in algebra they are required to recognize the structure that they have been able to avoid in arithmetic. She also mentioned (p. 43) that in contrast to the large number of papers on students' concepts of variable, relatively little research had addressed itself specifically to the concept of an algebraic expression. Collis (1974), R. B. Davis (1975) and Booth (1984) pointed out the incongruencies between arithmetic and algebra and the consequent inability of novice algebra students to regard algebraic expressions as legitimate answers. Herscovics and Chalouh (1985) noted examples of cognitive conflict created by the existence of both an arithmetical and an algebraic frame of reference in the mind of novice algebra students.

The popular belief that algebra should be introduced as growing out of arithmetic was challenged by Lee and Wheeler (1989). They gathered test-interview data which showed that such an approach faced serious pedagogical difficulties and might obscure the genuine obstacles. It was found that for many students (even some of those who were successful at standard algebraic tasks) algebra and arithmetic were two dissociated worlds; when these students were confronted with both, their arithmetic appeared disturbed by algebra.

The present paper owes various ideas to the reports of two British projects: CSMS ("Concepts in Secondary Mathematics and Science") and SESM ("Strategies and Errors in Secondary Mathematics"), see Hart (1981), Booth (1981, 1983–84, 1984), Küchemann (1981). These studies found that:

– some kinds of errors are widespread among students of different ages, independent of the course of their previous learning of algebra,
– surprisingly, the differences between skills of 13-year-olds and 15-year-olds were not as big as expected.
– a significant number of students did not use the formal methods taught in schools, preferring some intuitive strategies.

The students' procedures often referred to objects of real life (e.g., the algebraic expression $8a$ was interpreted as short for '8 apples'). Such procedures were efficient in the case of simple tasks, e.g., transforming $2a + 3a$, and were categorized as lower forms of understanding; they were not sufficient for somewhat more difficult tasks, e.g., $3a - b + a$. Such low-level procedures were used by both younger and older students, and this partially explains (Booth, 1983–84) the lack of definite differences in their algebraic skills.

METHOD

The study included all 108 students from 4 classes during Grades 7 and 8 in 3 schools in Gdańsk and Malbork. All teachers involved in the study had a university degree with some postgraduate training. The topics followed the Polish national curriculum. No attempt was made to implement any modifications or novelties. The author had taught one of the classes from Grade 4 on and gathered documents: plans of lessons, notes on the childrens' behaviour, copies of students' papers from tests.

In the middle of Grade 7 the students of all these four classes were given paper-and-pen Test I consisting of three problems. The purpose of Problem 1, divided into 10 independent items, was to provide information on how the children would transform the given algebraic expressions. The selection of the structure of each expression was based on observation of difficulties encountered by students. Problem 2 concerned their understanding of the value of an expression for a given value of x and of the equality of two expressions. The style of wording of the problems followed that in textbooks used by the children.

Problems given in Test I for Grade 7

Problem 1. *Write each of the following expressions in the simplest form possible*:

(a) $6x + 3x$, (b) $6x \cdot 3x$, (c) $3x - 6x$,
(d) $3x \cdot (-6)$, (e) $6x : 3$, (f) $-3 + 6x$,
(g) $2x + 3 - 3x$, (h) $-x + 2 - x^2 + 1$, (i) $(6x + 3x)^2$,
(j) $2x^2 - x - 5x^2$.

Problem 2. *Find the numerical value of expressions* (g) *and* (h) *in Problem* 1 *for* $x = -5$.

There was no time limit; students returned their work when they felt they had finished (yet, nobody exceeded the duration of a standard lesson, i.e., 45 minutes). The author analysed the responses, tentatively classified the errors, and then selected a sample of 51 students for oral interviews, which were organized within two weeks of the test. All care was taken to ensure that (i) the sample included the largest possible variety of errors and childrens' strategies and (ii) it was a reasonable cross-section of students' ability levels from poor to very good.

Each of the 51 children was interviewed individually by the author. The interview lasted from some 20 minutes to 45 minutes. The child received his/her paper (which looked exactly the same as it did when the child had returned it, without any marks or corrections), was asked to read it carefully and was invited to make possible corrections with a coloured pen. When this was done, the child was asked to explain how he/she obtained the results. When Problem 2 was discussed, each student was asked two additional questions. First: "*Which version of the expression (the original or the simplified one) did you choose for computing the value of the expression*" (or: "*would you have chosen*" if the child had not computed it). Second: "*Suppose you substitute the given number in the other version of the expression; will you get the same value?*".

At the end of Grade 8 the same four classes were given Test II and the same children were subject to analogous interviews. The problems for Test II had the same purpose as those in Test I, but the choice of the expressions reflected the students' progress. Their relative difficulty compared with the curriculum requirements remained much the same.

Problems given in Test II for Grade 8

Problem 1. *Perform the operations*:

(k) $(-2x) \cdot 8x$, (m) $2x : 8$, (n) $8x^2 : 2x$,
(o) $-2x^2 + 8 - 8x - 4x^2$, (p) $(-4x + 3) + (-1 + 2x)$,
(q) $(-4x + 3) - (-1 + 2x)$, (r) $(-4x + 3)(-1 + 2x)$,
(s) $-2(3x - 8)$, (t) $2x(3x - 8)$, (u) $(3x - 8) : 2$,
(v) $(8x - 2x)^2$, (w) $(8 - 2x)^2$,
(y) $(2x^2 + 5x) - 3x$, (z) $(12x^3 - x^2) - 3x(2x + 1)(2x - 1)$.

Problem 2. *Find the numerical value of expression* (z) *in Problem* 1 *for* $x = -3$.

Each test resulted in: a) about 100 students' papers, b) protocols from about 50 interviews. This material was analysed by the author seeking for patterns and regularities concerning the following questions: *What is the number and the percentage of correct solutions for each part of the test and for each student? What is the hierarchy of relative difficulties of the expressions to transform (measured in terms of correct solutions)? What kind of errors were made by students? What procedures were used by them? Can we divide them – according to their results – into some characteristic groups?* The work on the data was confined to (i) qualitative and quantitative analysis of the students' errors and their difficulties and (ii) qualitative analysis of the procedures used by the students. This paper deals with the latter. Problem 3 (concerning inverse operations) is not discussed here.

DESCRIPTION OF TYPES OF PROCEDURES

There were 10 expressions to simplify in Test I and 14 in Test II. Thus, altogether, the data concerned about 2400 written transformations and notes of oral descriptions of about 1200 transformations made during the interviews. As a result of the search for patterns, seven types of students' procedures have been identified and labelled as follows: (A) *Automatization*, (F) *Formulas*, (GS) *Guessing-Substituting*, (PM) *Preparatory Modification* of the expression (this includes a subtype: *Atomization*), (C) *Concretization*, (R) *Rules*, (QR) *Quasi-rules*. They are explained below.

The author has found that these 7 types suffice to describe over 90% of the twelve hundred cases of the behaviour of the students who transformed the expressions during the interviews. The remaining cases appear too vague to classify; procedures of some students were borderline cases.

Still, only part of those 90% can be described in terms of a single type of procedure; there may be several (usually two or three) types recognized in the way the child transformed an expression; characteristic combinations were: (PM)+(R)+(C) and (R)+(GS). This should be understood as the use of several types in consecutive steps of a transformation rather than as one mixed type. Some types, e.g., (GS) or (PM), appeared always in conjunction with another type whereas some other types, e.g., (R) or (A), often appeared as pure types. A child might use a procedure of one type for some expression and of another type for another expression.

The types should not be interpreted in terms: correct–incorrect. A student might use a procedure of type *Rules*, say, in a mathematically correct way in one problem and then use a wrong rule in another problem. Also procedures are not evaluated here as 'better' or 'worse'. They are just described and their most important features are identified.

(A)-procedures (that is, procedures of type (A) *Automatization*) were used by students who were most successful at the tests, those of type *Quasi-rules* – by least successful. An account of findings on correlations between the types of procedures and students' achievements is given below.

(A) Automatization

The characteristic feature of type (A) is automatization of the operations involved in the transformation: the student immediately knew the correct result of the transformation and was genuinely surprised by the questions "*How did you get the result?*" and "*Why do you think it is correct?*". Typical answers were: "*It is obvious*", "*I have no idea why I know it*", "*Perhaps we had this in the class, but I do not remember*", "*I just know it*", "*I see it at once, just looking at the expression*". The student behaved as if the operations were somehow interiorized and became part of his/her inner nature. However, when the child performed the operations skilfully and either (i) he/she was able to explain them or (ii), in contrast, it appeared that he/she only reproduced what had been taught ("*I learnt this way during my classes*"), the procedure was not identified as (A).

Type (A) is a very special type of procedure. One may object to the use of the word 'procedure' when the child's reasoning is not disclosed. Yet, an (A)-procedure does satisfy the conditions formulated in the Introduction. It is an act of transforming the expression and has certain characteristic features. During the interviews the author sometimes had doubts how to classify certain procedures, but this never happened in the case of (A)-procedures. It should be emphasized that the crucial criterion was the behaviour and attitude of the child, not only specific words used during the interview; consequently, it is difficult to present it faithfully in print. Type (A) was noted only in Grade 8, only in the case of best students, and only when operations were really easy for them. It never occurred when operations on negative numbers were involved.

(F) Formulas

(F)-procedures are characterized by the use of formulas with variables; the child should be able to substitute a number or a term (such as $3x$ or $-5x^2$) in the formula. For instance, dealing with $(8x-2x)^2$ in (v) a student said: "*I used the formula* $(a+b)^2 = a^2+2ab+b^2$*;* a *is* $8x$, b *is* $-2x$"; another student, explaining a transformation of $(2x+1)(2x-1)$ in (z), said: "*there is such a formula:* $(a+b)(a-b) = a^2-b^2$". A student explained the transformation of $(-4x+3)(-1+2x)$ in (r) by saying "*I used the formula of short multiplication*"; after the question: "*What formula do you mean?*" he wrote: $(a+b)(c+d) = ac+ad+bc+bd$. Such formulas were shown in the classroom many times. Purely oral explanations such as "*I multiply each term in the first parentheses by each term of the second parentheses*" are not identified as type (F), but as an essentially different type (R) *Rules*, described below. Many students could give an oral description of their procedures but were not able to write pertinent formulas with letters (when they were asked to write a formula, they repeated the oral description).

Part of (F)-procedures lead to incorrect results; students either used erroneous formulas, e.g., $(a-b)^2 = a^2-2ab-b^2$, or used correct formulas in a wrong way, e.g., used the formula $(a-b)^2 = a^2-2ab+b^2$ and wrote: $(8-2x)^2 = 8^2-2\cdot 8\cdot(-2x)+(-2x)^2$.

(GS) Guessing-Substituting

Students using a (GS)-procedure had some idea what the result of the transformation should look like and then 'checked it with numbers', that is, substituted a number (most often it was 2 or 3) in the original version of the given expression and then in the new one. Sometimes they also tried another number. The purpose of the substitution was to examine the equality of the two expressions or to check whether the guess was correct. Some students commented, e.g., "*I substitute a concrete number here to check if this is all right. No, I don't do this always, only when I don't know or am not sure. I remember how to transform some expressions, for instance,* $3x$ *plus* $6x$". Type (GS) was seldom used in Problem 1. Part of the procedures identified as (GS) gave correct results, part – wrong. The errors usually resulted from the lack of skill in performing the operations.

(PM) Preparatory Modification

Procedures identified as *Preparatory Modification* of the given expression were used by students who aimed at changing the given surface structure of the expression to a more elementary or easier-to-understand form of it. E.g., a child pointed the expression $-2x^2+8-8x-4x^2$ in (b), said: "*I'll change it to addition*" and wrote $-2x^2+8+(-8x)+(-4x^2)$; another child uttered "*This is the sum of* $-x^2$*,* 8*,* $-8x$ *and* $-4x^2$". Dealing with the subtraction $(-4x+3)-(-1+2x)$ in (q) some students said "*I am changing this to addition*" and wrote: $(-4x+3)+(-1)(-1+2x)$; however, purely verbal explanations were more frequent, e.g.: "*This minus sign before the second parentheses is the same as if it were* -1 *here*". A student said: "*In the second parentheses I changed the signs to the opposites*" and wrote: $(-4x+3)+(+1-2x)$.

This approach was often (particularly in Grade 8) commented on by the children, e.g., "*It pays to replace by addition; addition lets you move freely*". Sometimes a student who found an error in his/her paper, e.g., in $-2x^2 + 8 - 8x - 4x^2$ in (o), uttered angrily: "*If I had written this as a sum, I would not have made this mistake*". Some students tried to order the terms according to powers of x, explaining, e.g.: "*to make it easier*" or "*I want to have the like terms together*". One child changed the sum $-2 + 8x$ to a safer $8x + (-2)$.

In the preliminary analysis, a type called *Atomization* was distinguished. During the interviews, some seventh graders wrote, e.g.,

$(*)\ 6x + 3x = x + x + x + x + x + x \quad + \quad x + x + x;$

this helped them to see that it was $9x$. Nevertheless, having analysed the interviews with eighth graders the author decided to regard *Atomization* as a special case of (PM) *Preparatory Modification*.

Students used (PM)-procedures in the case of products too, in some characteristic ways. E.g., a kind of atomization consisted in explicitly writing the multiplication symbol in each monomial, e.g., while dealing with $6x \cdot 3x$ some students wrote $6 \cdot x \cdot 3 \cdot x$ and explained: "*I am filling in the dots and then I can see I have only multiplication, and with multiplication the numbers may be transposed*."

Changing division to a fraction is also regarded as a (PM)-procedure. For instance, in (n) the division $8x^2 : 2x$ was first replaced by $\frac{8x^2}{2x}$ because "*It is convenient to change to the fraction bar. You can then simplify*". In the same task (n) another student replaced the numerator of the fraction $\frac{8x^2}{2x}$ by $8 \cdot x \cdot x$ and said: "*I am writing out x^2 as x times x. Now you can more easily see what can be cancelled*"; this is again a kind of atomization. Sporadic seventh graders wrote x^2 in the expression $2x^2 - x - 5x^2$ in (j) as $x \cdot x$ or uttered "*x times x*", explaining "*this is to avoid mistakes and incorrect reductions, to see that it is x times x, not just x*".

Another kind of a (PM)-procedure was converting division to the multiplication by the inverse number, e.g., a student wrote $6x : 3 = 6x \cdot \frac{1}{3}$ and declared: "*Now I have multiplication only, so I know what to do*".

Most of the (PM)-procedures led to correct results; moreover, they were reasonable as preparation to the next step of the transformation.

Some of the (PM)-procedures, e.g., decomposition $(*)$ or writing x^2 in a polynomial as $x \cdot x$, had never been shown during any lesson.

(C) Concretization

(C)-procedures are described as follows: The student imagined some model of the abstract operation to be performed (often treated as analogy, with characteristic phrases such as "*the same as*", "*similarly to*"). The model might be somewhat abstract, but still more concrete than algebra. Often it referred to relations of concrete objects in everyday life, e.g., "$3x$ *and* $6x$ *make together* $9x$ *in much the same way as* 3 *apples and* 6 *apples make together* 9 *apples*" or "$6x$ [divided] *by* 3 *is* $2x$, *because if* 6 *apples are shared by* 3 *people then each will get* 2 *apples*".

Another, also frequent, kind of concretization consisted in treating x as if it were something concrete, but without referring explicitly to external objects, e.g., "3 x*'es and* 6 x*'es make* 9 x*'es*", or "6 x*'es are divided into* 3 *equal parts; there will be* 2 x*'es in each part*".

Only one student (out of fifty) referred to possible numerical values of x: "$6x$ *and* $3x$ *is* $9x$, *as for instance* 6 *times* 3 *and* 3 *times* 3 *is* 9 *times* 3", though such explanations had been frequent during lessons (at least in the class taught by the author). Only one student used the number line.

No student ever used an adequate concretization leading to a incorrect result. However, there were sporadic inadequate concretizations, e.g., while $6x$ was explained as 6 cherries, $6x \cdot 3x$ became '18 squared cherries'.

(R) Rules

A student's procedure is categorized as a procedure of type (R) *Rules* when the following conditions are satisfied:

(1) the student referred to a rule describing a way of transforming the given expression (saying, e.g., "*It's a rule*") and/or indicated (either explicitly or implicitly) the way of doing this so that it was clear how to formulate such a rule;

(2) it was not a procedure of any of types (F) *Formulas*, (GS) *Guessing-Substituting*, (PM) *Preparatory Modification* or (C) *Concretization*;

(3) no inconsistency was ever observed in the student's behaviour (otherwise the procedure was categorized as type (QR) *Quasi-rules* described below), i.e., each time when the expression was of the same structure (with different coefficients) the child used the same rule or consciously chose another way though was aware of the applicability of the previous rule.

Some children first referred to a rule and then, dealing with an analogous expression, said laconically, e.g., "*Well, I am computing*" or "*I am transforming*". Yet, after having been pushed to be more specific, almost always their explanations concerned the same rule. Few students were not able to explain their way and then the author suggested what might be the rule; the child either confirmed this or denied it and tried to correct the formulation. The consistency of the explanations (or its lack) was thoroughly traced during the interviews and was a basis for the decision whether to qualify the procedure as type (R) *Rules* or rather as (QR) *Quasi-rules*.

In most cases the rules were correct, verbalized in the form of how to proceed, often in the first person (a natural reaction to the way the question was phrased). It should be emphasized that the description of type (R) does not take into account whether the rule is correct or not; *if a wrong rule is used systematically and in a coherent way, the procedure is categorized as* (R). This stipulation is crucial to the present study.

An example of a rule leading to a correct result in $6x + 3x$ in (a) is: "*In addition, you add coefficients and copy* x". An apparently similar rule, applied to $6x \cdot 3x$ in (b), yields an incorrect result; a child performing the multiplication $3x \cdot 6x$ said: "*I must multiply these numbers* [showed the coefficients] *and then write* x". The students who used such a rule in (b) often reacted to expression $3x \cdot (-6)$ in (d) by saying: "*This can't be done, because these terms are not similar*"; still, the author regards such behaviour as consistent, for the structure of (d) differs from that in (b).

Some of the rules were presented by students in much the same way as the teacher had done it, e.g., "*When we subtract an algebraic sum, we have to replace the sign of each term by its opposite*". However, quite often the rules quoted by students appeared to have been formulated by themselves as a result of their experience in the classroom. This was obvious in the class taught by the author, but the teachers of the three other classes expressed the same opinion. Sample: "*When we subtract similar expressions, we subtract the coefficients and the letter remains the same*".

Students' modifications of rules were particularly striking when they were not correct. Samples: in (j), "$2x^2 - x - 5x^2 = 4x - x - 25x = -22x$ *because we must first compute the powers here*"; in the case of (q) a student commented $(-4x + 3) - (-1 + 2x)$ by saying: "*we change the sign of the first term, that is, it's* -1*, and the other term is copied without change*"; dealing with $(-4x + 3)(-1 + 2x)$ in (r) a student said: "*we multiply the similar terms, that is,* $-4x$ *times* $2x$ *plus* 3 *times* -1".

According to a recommendation in the curriculum, the author had kept explaining the algebraic transformations to her class by referring to the basic properties of operations. Yet, this kind of explanation was rare during interviews with those children, though they were pushed to do so. Moreover, when students did refer to such a law, it was often incorrect. E.g., a child wrote: $3x + 6x = 9x$ with explanation: "*in virtue of the associativity law*"; another student wrote $(8 - 2x)^2 = 8^2 - (2x)^2$ and commented: "*by distributivity law*". The only properties that were correctly quoted by students were: 1) distributivity in the case where $a(b + c)$ was transformed into $ab + ac$, 2) commutativity (mostly in the case of addition, sporadically – multiplication). References to associativity were very rare.

The students rarely attempted to justify their rules (even after an explicit question "*Why do you think it works*"), saying e.g.: "*This was the way we did it in the classroom*" or "*Our teacher taught us so*". Such answers were also given in cases where for sure such rules had never been formulated during any lesson. A student of the class taught by the author worked with $6x \cdot 3x$ and said: "*when we multiply* $3x$ *by* $6x$*, we multiply* 3 *by* 6 *and copy* x"; his explanation was astonishing: "*You told us so*".

(QR) Quasi-rules

This type of procedure is characterized as follows: The students quoted a rule (or it was easy to identify the rule after listening to the explanation) but did so in an inconsistent way. Some of these transformations were correct, a lot were not. However, the author's main point is that the difference between frequencies of errors in (R) *Rules* and (QR) is of secondary importance. What is crucial is the consistency of the student's system of beliefs. (R)-procedures were consistent, (QR)-procedures were not.

For instance, dealing with expression $-2x^2+8-8x-4x^2$ in (o) a student said: "*First we must raise to the power*" and replaced the first term $-2x^2$ by $4x^2$; a moment later he copied the last term $-4x^2$ (without 'raising to the power'). Another student, dealing with $-4x + 5 + 3x$, explained: "*We can only collect* $-4x$ *with* $3x$*,* 5 *is not like to them*" and immediately afterwards 'simplified' all terms of the expression. Even when the same expression was transformed once again, some students first claimed what rule they used and a moment later uttered another (non-equivalent) rule, as if 'rules' were something arbitrary. Moreover, for some students the fact that they obtained different results did not appear to be a problem, e.g., a student replaced x^2 first by x, then by $2x$, and next time by $2 + x$. Their performance seemed to be quite haphazard, the computations were fraught with unsystematic errors. (QR)-procedures appeared to be *symptoms* (rather than reasons) of serious troubles. The author had the impression that these students formulated the rules *ad hoc*, recalled something, uttered a rule, forgot it shortly afterwards, invented another rule which was to cover the situation, and so on. E.g., a student wrote $6x \cdot 3x = 18x$ with a comment: "*in multiplication I multiply numbers and write* x" and a moment later he had no idea what to do with $9x \cdot 9x$. When such students were asked "*Why do you think so*", very often they withdrew and asked, e.g., "*Oh, isn't it?*" or quoted another rule and claimed that it was 'really' the rule that was used. Several of them appeared deceived by 'attractors', that is, superficial features which attracted their attention, e.g., when they saw $+3$ at one place and -3 in $-3x$, they were tempted to 'cancel', without understanding the structure of the expression.

DEVELOPMENT OF STUDENTS PROCEDURES

An analysis of the results of the study clearly shows that *students' procedures did evolve in the course of 15 months between Test I and Test II*.

The algebraic structures of some expressions from Test II were somewhat analogous to those of Test I, namely: (v) $\leftrightarrow$ (i); (m) $\leftrightarrow$ (e); (k) $\leftrightarrow$ (b); (p),(q) $\leftrightarrow$ (g); (y),(o) $\leftrightarrow$ (h),(j). The purpose of this arrangement was to get a chance of comparing the procedures used by the same student dealing with the same kind of expression after 15 months.

Having selected the expressions for Test II (but before administering it) the author believed that an analysis of procedures used in simplifying (k)–(z) would enable her to compare them with procedures that had been used for expressions (a)–(j): the reduction of similar terms in polynomials of degree 1 and the multiplication of px by a number, which were components of transformations needed for (k)–(z). This assumption turned out wrong, for the students who performed such a transformation as part of a complex task focussed on the latter. Auxiliary parts of the main transformation were carried out mechanically; during the interviews students spoke only about the prime task. Questions concerning these 'subroutines' distracted the interviewees; they were reluctant to respond. This does not mean that these procedures were classified as (A) *Automatization*. On the other hand, the change of natural coefficients in Test I to fractions or negative integers in some expressions (given or resulting) in Test II might force the student to change the type of the procedure (e.g., (C) *Concretization* might no longer work).

Procedures of type (QR) *Quasi-rules* were noted in both tests; their frequency in Grade 8 was a little lower than in Grade 7. Some students passed from (QR) in Grade 7 to (R) *Rules* or (F) *Formulas* in Grade 8 while some other students' procedures were categorized as (QR) also in Grade 8; sporadically, a student who used (R)-procedures in Grade 7 would regress to (QR) in Grade 8.

(R)-procedures were the most common type of procedures in Grade 7 (each student interviewed in Grade 7 used such a procedure at least once); in Grade 8 their usage was increased further. Though the expressions in Test II were more difficult than those in Test I, the percentage of correct (R)-procedures increased. Moreover, there were numerous indications that the rules used by many eighth graders formed a somewhat coherent system. For instance, a student said: "*I multiply this number by that number* [shows coefficients] *and x by x, but in addition you add the numbers and the letter remains unchanged*" or "*here we'll get x^2; there was a time when I thought it would be only x as it was the case with addition; now I know that in* *addition and subtraction it's this way and in multiplication – otherwise*". Some students who used only (R) in Grade 7 extended later the scope of their types of procedures and used also (C) *Concretization*, (F) *Formulas* or (PM) *Preparatory Modification*. See also Diagrams 1–2 below.

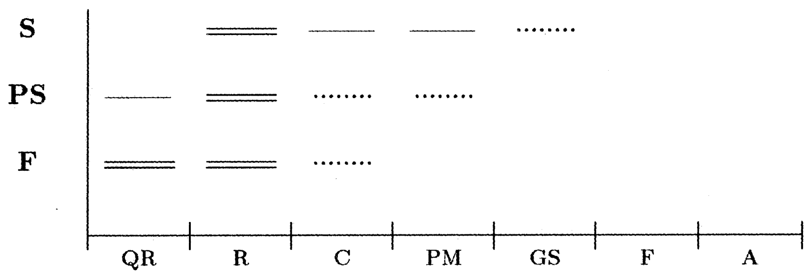

Diagram 1. Frequencies of types of procedures of the students of Grade 7 depending on their achievement level. Each segment of the horizontal line (at the bottom) represents one type of students' procedures; information provided in the column above the segment concerns this type. Each level at the vertical line represents one group of students; the corresponding row contains information about the frequencies of types of procedures used by students of this group. Double line ═══ denotes the prevailing type (for the group of students shown on the left). Continuous lines —— denote types which were frequent but not prevailing. Dotted lines · · · · · · ·· denote types which appeared only sporadically or were rare. Blank spaces denote types which were never observed.

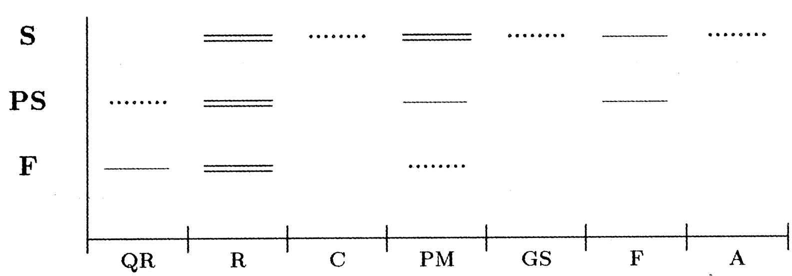

Diagram 2. Frequencies of types of procedures of the students of Grade 8 depending on their achievement level. For further explanations, see Diagram 1.

CORRELATION BETWEEN TYPES OF PROCEDURES AND STUDENTS' ACHIEVEMENTS

After each test the author distinguished three groups of students, labelled **S**, **PS** and **F**. Their description is based on the frequencies of correct transformations in Problem 1. In Grade 7, group **S** (short for ‘success’) consisted of students who got over 70% of possible points (i.e., they correctly transformed at least 8 expressions out of the total of 10 expressions, fractional scores were not given); group **PS** (short for ‘partial success’) consisted of students who had over 30% and at most 70% of correct transformations[(3)]; group **F** (short for ‘failure’) consisted of students who had at most 30% of correct transformations in Problem 1.

It was found that, approximately, *20% of students participating in Test I qualified for group* **S***, about 65% of students were in group* **PS***, and about 15% in group* **F**. The frequencies of types of procedures were correlated to the level of students’ success, as shown in the following diagrams.

In Grade 7, procedures of students in **S** were the most varied and those in **F** the least varied. Typically, a student in **S** used (R) *Rules* and one or two other types; students who used only (R) were rare in **S**. (R) was more frequent in **PS** than in **S**.

In Grade 8, a similar analysis of test and interview data concerning Problem 1 confirmed the existence of three distinct groups (determined by the same bounds: 70% and 30%): **S** consisted of students who wrote Test II and got at least 10 points (out of 14), **PS** of those who got from 5 to 9 points, **F** – those who got up to 4 points.

The change of distribution of scores in Test II was fundamental. The percentages were nearly doubled or halved: **S** – *over 35% of students* (much more than in Grade 7), **PS** *also over 35%* (much less than in Grade 7), **F** – *about 25%* (much more than in Grade 7).

Each individual student participating in Test I was traced to Test II. *Almost all students of group* **S** *(in Grade 7) remained in* **S** *(in Grade 8). Students in* **PS** *split into three parts: the largest part, consisting of about half of* **PS** *in Grade 7, remained in* **PS** *in Grade 8; a quarter of* **PS** *in Grade 7 advanced to* **S***, and another quarter dropped to* **F**. *Over half of the students from* **F** *remained in* **F***; other students in this group advanced to* **PS**. Let us note that the expressions in Test II were more difficult than those in Test I. Consequently, remaining at the same percentage level means notable progress of the student.

In Grade 8, procedures of students in **S** were even more varied than before. The combinations (R)+(PM)+(F) appeared quite often, sometimes together with one or two other types: (GS), or (A), or (C). Also in **PS** the combination (R)+(PM)+(F) was frequent. The patterns of change of procedures of individual students manifest the following general trends:

(1) Most of the students whose percentage of correct transformations increased from Grade 7 to Grade 8 used in Grade 8 more types of procedures

than in Grade 7. Most of the students who did not make progress used the same number of types (or sometimes less).

(2) Almost all seventh graders who used procedures of types (GS) *Guessing-Substituting*, (C) *Concretization* or (PM) *Preparatory Modification* belonged to either **S** or **PS**; almost all of them qualified for **S** in Grade 8.

SUBSTITUTING A NUMBER IN AN EXPRESSION

The report on the results of interviews concerning Problem 2 is restricted here to those aspects that shed a light on how the students understood the identical equality in the transformation. Three possible features of students' responses to Problem 2 were considered. First is the Elementary Understanding of Substitution (EUS) interpreted as substituting the given number instead of the variable at each place where the variable appears and including the number in the structure of operations. Second is the Correct Substitution and Computations, which comprises EUS and, additionally, performing correct computations. Third feature (independent of the above two) is SSVE, Substituting (at least once, correctly or not) in the Simplified Version of the Expression (g) or (h) rather than in the original one. In Grade 7, Problem 2 was correctly solved by 3 students only (out of 108); 5 students did not attempt it. Nevertheless, 85% of students (90% of those who tried to solve the problem) manifested EUS. Most errors concerned computations on negative numbers, which caused serious troubles. Only about a quarter of seventh graders manifested SSVE though it had been explained many times in the classroom. The remaining students used the original version (or did not attempt Problem 2).

During the interviews, the question: "*Suppose you substitute the same number $x = -5$ first in the original expression and then in your simplified expression. Will you get the same value?*" surprised many students. Samples: "*I've never thought of this*", "*I do not know if this would turn out the same; I never tried*", "*No, why, there are different numbers and different operations here and there*", "*I don't know, I substituted in the simplified version because we did so during our lessons*". The reluctance of many students to use the simplified version was conspicuous. The choice of the simplified vesion in Test I was generally a consequence of the classroom routine ("*Our teacher always says we should substitute in the simplified version*"). In most cases (but not in all) this choice reflected the awareness that the two substitutions (if no mistake was committed) must have given the same number. On the other hand, some students knew that the results should be the same and yet preferred the original version as safer.

In Grade 8, only about 80% of students attempted to solve Problem 2 (significant decrease from Grade 7) but the number of those who solved the problem correctly increased tenfold. EUS was evident in the work of almost all students who started the problem. The increase of the number of students who used the simplified version of the expression was considerable (from about 25% in Test I to about 75% in Test II, i.e., to almost 90% of those who attempted the problem).

CONCLUDING DISCUSSION

For the author, one of the most surprising results concerning Problem 1 was the presence of three groups of students (described above, labelled **S**, **PS**, **F**), originally distinguished by their scores, that is, by the *quantity* of errors, but differing – as it turned out – also in types of procedures (and in numbers of different procedures used by the students) and in the kinds of prevailing errors, that is, differing in *qualitative* features.

Comparing the results in Test I with those in Test II we get six major patterns of the development of students' performance, namely: passing from **S** in Grade 7 to **S** in Grade 8; from **PS** to **S**; from **PS** to **PS**; from **PS** to **F**; from **F** to **F**; and sporadically from **F** to **PS**. Only one student dropped from **S** to **PS**. The remaining two possibilities (from **S** to **F**, from **F** to **S**) were absent.

In Grade 7 students often used wrong rules, but many of them improved, in a long process, and after 15 months used only correct rules. It was very optimistic that three quarters of the most numerous group **PS** of Grade 7 (i.e., students with middle results), supported by careful and patient help of the teacher, could master rudiments of transformations.

It was found that *many students used procedures that had never been shown during any lesson and were absent in their textbooks*. Somehow these spontaneous procedures were natural for them. Moreover, students frequently referred to rules (often of their own invention), usually expressed in the form of actions to be performed; formulas were rarely used.

The rules depended on students' previous experience (particularly on the kinds of tasks) as well as on their personalities and types of mind; children taught together in the same class conceived diverse rules.

The most important finding of the study is *the need for discriminating between procedures of type* (QR) *Quasi-rules and incorrect procedures of type* (R) *Rules*. Both may result in the same error in a written test; the crucial difference, which may be detected during a conversation with the student, is that (R)-procedures are consistent, i.e., the student uses the same procedure while transforming analogous expressions and, moreover, believes (or is even convinced) that the rule is the right one. In contrast to the above, (QR)-procedures are not consistent, often formulated *ad hoc*, and changed haphazardly. There is also an essential difference between the prognostic values of the two types of procedures: most of the students who used (R)-procedures, even erroneously, made progress and qualified for a higher group while students who used (QR)-procedures were much more likely to remain at a low level.

Results of the study seem to suggest that the emergence of incorrect (R)-procedures may be *a normal stage in students' development*. Many students of medium abilities go through such a stage. Perhaps it is as normal (and similarly surprising) as, e.g., the lack of conservation of the cardinal number by 6-year-olds, a well known discovery of Piaget. If students are helped to overcome the difficulties, they can progress and gradually learn to use correct rules; if not, they are in danger of complete failure.

Students' procedures did change during the 15 months separating the tests (the change was partly caused by learning new transformations). In particular, the frequency of procedures of type (C) *Concretization*, i.e., treating symbols as if they were concrete objects, decreased markedly. At the same time, an increase of procedures of type (PM) *Preparatory Modification* was conspicuous; they were directed towards grasping the structure of the expression (e.g., replacing differences by sums with negative coefficients, replacing quotients by suitable products or fractions). During the interviews, particularly in Grade 8, (PM)-procedures were used by students fairly often and spontaneously. *These procedures are potentially very good*, but they are unwarrantedly neglected by teachers and not appreciated enough in textbooks.

Procedures of type (GS) *Guessing-Substituting* are of limited use when a new kind of transformation is introduced. Their appropriateness and didactical significance are doubtful when they serve as a justification for the validity of a transformation. They could be more suitable for helping children rectify an error; yet, the interviewees used (GS)-procedures mainly to ascertain certain forgotten details, e.g., a student hesitated whether $6x \cdot 3x$ is $18x$ or $18x^2$ and checked which version was the right one. It seems worthwhile to learn more about the possible advantages of such procedures, particularly for fostering the understanding of the equality of two expressions.

The type (R) *Rules* is heterogeneous and probably one could distinguish certain subtypes in it. The same remark applies to the types: (PM) *Preparatory Modification* and (C) *Concretization*.

Comparing the results concerning procedures with previous research

Observations made in this study confirm previous accounts (Booth, 1983–84; Ćwik, 1984; Lee and Wheeler, 1989) of situations where students used their own procedures rather than those taught in school. In particular, many students referred to rules different from those explained and used in the classroom and, moreover, often did not distinguish their own rules from those shown by the teachers; some students even claimed that they had learnt their wrong rules in school.

Correct rules invented by students are rarely described in print; specific examples usually concern wrong rules, e.g., $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$. If such an error is mentioned in a paper, it is often difficult to figure out how to interpret the child's procedure: as *Quasi-rules* or incorrect *Rules*. The findings of the study *show that Krygowska's statement quoted in 'Background' is an oversimplification of the problem*. They also contradict the opinion of Fischbein (1994: p. 242) that in order to overcome errors (such as wrong cancellation of an algebraic fraction) the student has to understand the formal basis (definitions and theorems) that justifies algorithms. On the other hand, the findings of the study agree with the conclusion of Kirshner (1995) that *successful students tend to go through a phase of overgeneralizing distributivity before achieving fluency in manipulative skill*.

Procedures of type (C) *Concretization* identified in this study were similar to those found in the British projects CSMS and SESM. However, in contrast to the approach popular in England, the teachers of three classes in the study (out of four) had not used this kind of explanation; most of the children who argued this way did it spontaneously.

Lee and Wheeler (1989) noted that of the 268 students asked about possible truth of an algebraic equality, only 10 made any attempt to 'check with numbers', that is, to use some kind of *Guessing-Substituting*; a rule-bound approach (correct or not) was more frequent. They also found that the algebraic work was often considered by students on a par with a single numeric example or (in a few cases) rather less reliable.

The role of basic properties of operations in learning transformations

In Polish textbooks (and also in many textbooks used in other countries), the explanations of how to transform algebraic expressions most commonly refer to basic properties of operations. This should be contrasted with the well-known fact that students learning rudiments of school algebra encounter serious difficulties in applying these properties. Wierzbicki (1970) examined results of nation-wide assessment of results concerning a considerable sample of Polish seventh graders (one class of each of 490 schools selected for the study); he reported that only less than a quarter of students could identify the property of operations used by them in given transformations, while correct results were obtained by about half of the students. Research of Skałuba (1988) has confirmed the known fact that for many students, even in upper secondary school, application of an identical equality (i.e., use of a procedure of type *Formulas*) is really hard. Nevertheless, in several textbooks, properties of operations are explained in terms of formulas only, without other arguments.

The main reason why the list of basic properties of operations is so stable over years in various curricula is that, in fact, these properties are standard axioms of a *commutative ring with unit* (or, if the additional postulate of existence of division is included, one gets axioms of a *field*). Therefore these properties are so natural and obvious for the average mathematician that it is hard to conceive that they may be a cause of serious didactical difficulties and need not be a good starting point for school algebra. It is not well known that these axioms were not appreciated by mathematicians before the 19th century. Advocates of the 'new math' movement in the 1960's stressed the importance of these properties for mathematics education (as part of the program of developing thinking in terms of *structures* on sets); yet, it should be noted that these axioms had appeared in curricula much earlier.

The present study provides strong evidence that *referring to basic properties of operations is not effective*; the students were reluctant or not able to make use of these properties. Only sporadically did students ever refer to them. This feature is particularly striking when confronted with two facts: (1) these basic properties are recommended by Polish curriculum and appeared in the students' textbooks, (2) the teachers who taught the four classes in the study systematically explained and used these properties.

The role of geometric interpretation

In many countries (including Poland) school curricula advise teachers to use well-known geometric interpretations of algebraic transformations, e.g., the distributivity may be illustrated with suitable pictures of rectangles. Some authors (Sawyer, 1964; Bruner, 1966; Ruthven, 1989) described how to interpret quadratic polynomials with, e.g., wooden blocks: a small cube, a 'long' (a rode) and a 'flat' (large cuboid with square base) represented 1, x and x^2, respectively. Chalouch and Herscovics (cf. C. Kieran, 1989: p. 43) tried to use similar representations to help children develop meaning for expressions such as $2a + 5a$.

In the present study only one interviewee (out of fifty) ever referred to geometric interpretations, though they were present in their textbooks and the teachers had used such arguments during lessons.

Correctness of the students' procedures

While the result of a transformation of an algebraic expression is either correct or erroneous, it is much more difficult to say something definite about the correctness of the children's procedures. Some of them obey the requirements of rigorous thought and are unquestionable; some other procedures are plainly wrong. However, most of the procedures cannot be easily included in either category; they may be regarded as correct, but cannot be called "performing the operations in accordance to strictly applied rules". This casts doubt on the adequacy of popular phrases such as "proper mathematical methods" opposed to children's own methods. Dividing students' procedures into two classes: "formal" and "intuitive" is a false dichotomy (the former often connotes: "routine algorithm" or "a school method"). Still worse is thinking in terms of the dichotomy of "deriving complex transformations from the basic properties" and "practicing algebraic rules in a quite mechanical way".

Procedures of type (C) *Concretization* may be regarded as intuitive ways of explanation; they are of limited use but in the present study almost always lead to correct results. Procedures of type (PM) *Preparatory Modification* are mathematically sound, but usually served only as a preparatory step before using another procedure. Correctness of a procedure of type (F) *Formulas* can easily be judged by a mathematically competent person. Correctness of a procedure of type (R) *Rules* is much harder to judge.

Procedures of type (GS) *Guessing-Substituting* appear groundless, fostering bad habits, not compatible with correct mathematical reasoning. Checking with one or two numbers is not a substitute for a proof that the expressions are identically equal. Yet, one cannot say that such a procedure is always incorrect; it is occasionally used by mathematicians when they know the form of the target expression but coefficients are not certain.

Syntactic and semantic aspects of algebra

Many people believe that algebra should stem from arithmetic and that rationale and justification of transformations of algebraic expressions should be based on understanding analogous arithmetical transformations. The findings of the study do not confirm this belief; the problem is much more complex than such a simple statement.

Tall and Thomas (1991) and Gray and Tall (1993) pinpoint one of the most significant aspects of the problem: those children who can interpret arithmetic symbols only as operations to be carried out are confused by algebraic expressions such as $3a + 4b$ (see also Herscovics, 1989; Sfard and Linchevski, 1994). Lee and Wheeler (1989) conclude their study with a remark that the track leading from arithmetic to algebra is littered with procedural, linguistic, conceptual and epistemological obstacles. They also quote Filloy and Rojano as having suggested that the challenge of dealing with the syntax of an unfamiliar algebraic language tends to destabilize students' semantic control of the familiar arithmetical language.

The significant problem of distinguishing semantic and syntactic aspects of school algebra, considered by several authors (see, e.g., Kaput, 1989; Booth, 1989), is only touched on in this paper.

Authors of books on abstract algebra point out that expressions (such as, e.g., polynomials) may be viewed in two different ways.

First approach: One refers to the meaning of the expression regarded as a function of a variable, say x. The letter x stands for an arbitrary number. If a specific number is substituted instead of x, we get a number, called the value of the expression at this specific x. Two algebraic expressions are regarded as equal if their values are equal for each x (this definition has to be suitably augmented to cover the cases where the domains of the two expressions are different, e.g., because of zeros in a denominator).

Second approach: Algebraic expressions are considered as part of a formal system (a ring of polynomials, say), satisfying certain axioms (basic properties of operations). The meaning of an expression is not taken into account. For instance, the letter x in a polynomial $a_0x^n + \ldots + a_n$ has no meaning; the only thing that matters is the sequence $a_0, \ldots, a_n$ of coefficients and the way such sequences are added and multiplied. Two expressions in such a formal system are regarded as equal if their equality can be derived (in a number of steps) from the axioms.

The first of these two aspects of an algebraic expression, referring to its meaning (based on the meaning of a variable as symbol of a number), may be labelled as *semantic*. Put differently, the semantic aspect concerns algebra understood as generalized arithmetic. The second aspect is *syntactic* as it refers to the way the expressions can be manipulated according to formal rules, regardless(4) of the meaning of the symbols.

Though children's procedures are far from a scientific level of algebra, we may trace semantic and syntactic features of five (out of seven) types of procedures (such a distinction semantic-syntactic does not seem relevant in the cases of the two extremes: (A) *Automatization* and (QR) *Quasi-rules*, each for different reasons). Procedures of type (PM) *Preparatory Modification* (particularly those of subtype *Atomization*) as well as procedures of type (GS) *Guessing-Substituting* refer to the arithmetic meaning of the operations, and hence they have some semantic aspect. Type (C) *Concretization* is semantic, but in a different sense. On the other hand, (F) *Formulas* and (R) *Rules* appear rather syntactic.

A natural question concerns a possible correlation between (i) the students' achievements and (ii) dominant features (semantic or syntactic) of their procedures. Some hints can be found in Diagrams 1 and 2. More successful students used semantic procedures more frequently than those less successful. Still, a comparison between frequencies of (F) *Formulas* in Grade 8 suggests another answer: good command of algebraic transformations is more likely when the student uses procedures of diverse types. Thus, some diversity of methods and some balance between semantic and syntactic aspects of school algebra is desirable.

Didactical implications

Knowledge of childrens' procedures identified in the study may be helpful in the search for more effective ways of learning algebraic transformations: mathematically sound and naturally accepted by students.

Teachers often observe that many students do not want anything but rules. These students can – with difficulty – remember the rules; they feel unequal to doing anything else. The postulate that students performing transformations should be conscious of the meaning and appropriateness of each step is convincing for mathematicians but the observations of students in the most numerous, middle group **PS** in the study indicate that it is hard to achieve. Their learning initially appears to follow a syntactic approach ("do so and so").

This should not be interpreted as advocating "skills before meaning". In fact, skills and meaning should somehow develop together and should reinforce each other. Each student somehow constructs the meaning of algebraic transformations, though it may be different from ours.

Construction of the child's knowledge does not follow from deductive reasoning, but from gathering experience, comparing results, generalization (here is where their own rules are born) and revising (possibly refuting) the previous rules. Constant verification of what the students do (enabling them to detect erroneous assumptions) is crucial. The study shows that explanations advised in curricula and books for teachers are not so effective as it is assumed. Conviction that a single very clear general exposition will work is an illusion.

This does not imply that algebraic rules should be introduced without much explanation. On the contrary, the teacher should use every opportunity for pertinent explanations; they should refer to the current work of pupils, to their doubts, questioning, conceptions and misconceptions, errors, spontaneous generalizations. Learning algebraic transformations by practising without attempt to generalize the children's experience is futile. Children cannot remember specific tasks, but can remember certain conclusions, especially those which surprised them by contrast to their prior beliefs. Many students not only wanted to know rules but also strove to use them in a correct and consistent way.

Transmitting algebraic rules by the teacher, memorizing them by students and practising them in a mechanical way is worthless.

The students' knowledge should be a result of suitable practicing; if students are deprived of opportunity to gather experience, they lack possibility of forming a stable basis for further learning. Very important is that the teacher should prepare sets of tasks of diverse types, suitable for developing the prior knowledge of students.

It is wrong to tell children general rules without suitable previous experience. Equally wrong is practising without recapitulation, without comparing methods and contrasting different results. *The most important is that children construct rules themselves*. These are not simply rules that teachers try to transmit. Insisting that children refer to *our* rules, foreign to their thinking (particularly, to basic properties of operations), is pointless.

NOTES

(1) A fuller report on the results of the study (including a detailed analysis of childrens' errors and an *a posteriori* hierarchy of difficulties of items of the tests) was presented as a Ph. D. Thesis "Rozwój procedur stosowanych przez uczniów klas V–VIII przy przekształcaniu wyrażeń algebraicznych" [Development of procedures used by students of Grades 5–8 transforming algebraic expressions], Pedagogical University, Cracow, 1994 (277 pages plus 144 pages of Appendix). Preliminary findings were presented at ICME-7 in Quebec (Demby, 1992). The author is greatly indebted to Professor Stefan Turnau, the supervisor of the Thesis, for his guidance, and to Professor Zbigniew Semadeni from the University of Warsaw for his help in preparing the English version of this article. Several persons have read the first draft of this paper and made various suggestions; the author is particularly grateful to Prof. Milan Hejný (Prague) and to Ms. Astrid Defence (Montreal) for their comments.
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(2) The quantitative data (percentages etc.) about procedures are not attempted here, because of the difficulty in determining the exact type of procedure in some cases. It is only mentioned whether – in various situations – certain types were prevailing (that is, most frequent, predominant), or were frequent, sporadic, rare, absent.

(3) The bounds: 30% and 70% were based on an analysis of the diagram of frequencies of the numbers of correct transformations obtained by all 108 students who had written Test I (in particular, on an analysis of how these frequencies were grouped). It was found out that these three groups of students were also characterized by the *prevailing kinds of students' errors*; this confirmed the choice of the bounds.

(4) However, Thom (1973) has asserted that, in practice, for the mathematician every statement that he/she is working with has some meaning regardless of how formally it is presented.

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