Ch. 6 Root Locus Analysis

6-1 Introduction

finding the roots of the characteristic eq. by W. R. Evans

6-2 Root Locus Plots

Angle and Magnitude conditions

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

Characteristic eq.

$1+G(s)H(s)=0$, or $G(s)H(s)=-1$

Angle cond. $G(s)H(s) = \pm 180^o(2k+1)$ $(k=0,1,2,\cdots)$
Magnitude conds. $|G(s)H(s)|=1$

The value of s that fulfill both the angle and magnitude conds. are the roots of the Characteristic eq. or the closed loop poles.

For example;

$$1+G(s)H(s) = 1 + \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} = 0 \quad , \qquad (K>0)$$

$$G(s)H(s) = \frac{K(s+z_1)}{(s+p_1)(s+p_2)(s+p_3)(s+p_4)}$$

$-p_1$, $-p_2$; complex conjugate

angle of $G(s)H(s) = \phi_1 - \theta_1 - \theta_2 - \theta_3 - \theta_4$

mag. of $G(s)H(s) = \dfrac{KB_1}{A_1A_2A_3A_4}$

Root locus; always symmetrical w.r.t the real axis

Examples 6-1

$$G(s) = \frac{K}{s(s+1)(s+2)}, \quad H(s) = 1$$

angle of $G(s) = \angle \frac{K}{s(s+1)(s+2)} = -\angle s - \angle s+1 - \angle s+2 = \pm 180^o (2k+1), \quad (k = 0, 1, 2, \cdots)$

magnitude conds. $|G(s)| = \left| \frac{K}{s(s+1)(s+2)} \right| = 1$

6-3 Summary of General rules for root locus construction

$$1+G(s)H(s)=1+\frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}=0 \quad , \qquad \{K>0\}$$

1. Locate the poles and zeros of $G(s)H(s)$ on the s-plane.
   The root-locus branches start from open-loop poles
   and terminate at zeros (finite zeros or zeros at infinity)

   no. of branches = no. of roots of the characteristic eq. (open loop poles)

2. Determine the root loci on the real axis

   - from a pole or zero to another pole or zero
   - if the total no. of real poles and real zeros to the right of the test point is odd, then the test point lies on the root locus

3. Determine the asymptotes of root loci

   angle of asymptotes = $\dfrac{\pm 180^o(2K+1)}{n-m}$ $\qquad (k=0,1,2,\cdots)$

   n = no. of finite poles of G(s)H(s)
   m = no. of finite zeros of G(s)H(s)

4. Find the breakaway and break-in points

   Characteristic eq. $B(s)+KA(s)=0$

   breakaway or break-in points are roots of $\dfrac{dK}{ds}=-\dfrac{B'(s)A(s)-B(s)A'(s)}{A^2(s)}=0$

   not all roots of the above eq. are breakaway or break-in points.

5. Determine the angle of departure (angle of arrival) of the root locus
   from a complex poles (at a complex zero)

   Angle of departure from a complex pole =
   180^o - (sum of the angles of vectors to a complex pole in question from other poles)
   + (sum of the angles of vectors to a complex pole in question from zeros)

   Angle of arrival at a complex zero =
   180^o - (sum of the angles of vectors to a complex zero in question from other zeros)
   + (sum of the angles of vectors to a complex zero in question from poles)

6. Find the points where the root loci may across the imaginary axis
   Apply Routh's stability

7. Taking a series of test points in the broad neighborhood of the origin of the s plane
   around origin or jw axis

8. Determine the closed loop poles

* Comments on the root-locus plots

(1)

$$G(s)H(s) = \frac{K(s^m + b_1 s^{m-1} + \cdots + b_m)}{s^n + a_1 s^{n-1} + \cdots + a_n} \quad , \ (n \geqq m)$$

if $n \geq m+2$, a_1 is negative sum of the roots and independent of K

as K increased, some roots moves on the locus toward the left,

the other roots must move toward the right.

(2) Cancellation of poles $G(s)$ with zeros of $H(s)$

$$\frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2) + K(s+1)}$$

ch. eq. $[s(s+2)+K](s+1) = 0$

However,

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{s(s+1)(s+2)}$$

$$= \frac{s(s+2)+K}{s(s+2)}$$

reduced ch. eq. $s(s+2)+K = 0$

** canceled poles of $G(s)H(s)$ are closed loop poles.

6-6 Root locus analysis of control systems

* Orthogonality of root loci and constant gain loci

$|G(s)H(s)| = \text{const.}$

$\angle G(s)H(s) = \pm 180^o(2k+1)$

in GH plane, orthogonal to each other

So, in s-plane ,

constant phase (root loci) and constant gain root locus are orthogonal to each other

* Conditionally stable systems (Fig. 6-33)

* Non-minimum phase systems

$$G(s) = \frac{K(1 - T_a s)}{s(Ts+1)} \quad (T_a > 0), \quad H(s) = 1$$

Angle condition;

$$\angle G(s) = \angle - \frac{T_a s - 1)}{s(Ts+1)}$$

$$= \angle \frac{K(T_a s - 1)}{s(Ts+1)} + 180^o$$

$$= \pm 180^o(2k+1), \qquad (k = 0, 1, 2, \cdots)$$

=> $\angle \frac{K(T_a - 1)}{s(Ts+1)} = 0^o$

Stable if $K < \frac{1}{T_a}$.