Ch. 5 Transient and Steady-State Response Analysis

5-1 Introduction

Performance comparison
test signal

Typical test signal;
- step function
- ramp function
- acceleration function
- impulse function
- sinusoidal function

gradually changing function of time -> ramp function
sudden disturbance -> step function

Transient response and steady state response;

transient response; from the initial to the final
steady state response; as t approaches infinity

Absolute stability, relative stability, steady state error;

absolute stability; stable or unstable
eventually comes back to its equillibrium state
critically stable; oscillations of the output continue forever
unstable; output diverges without bound

relative stability;
steady state error;
output of the system at steady state does not exactly agree with he input

Classifications of industrial controllers

- self operated controllers
- two-position or on-off controllers; differential gap, cut-in, cut-out
- proportional controllers
- integral controllers

- proportional plus integral controllers

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt$$

$$\frac{U(s)}{E(s)} = K_p(1 + \frac{1}{T_i s})$$

- proportional plus derivative controllers

$$u(t) = K_p e(t) + K_p T_d \frac{de(t)}{dt}$$

$$\frac{U(s)}{E(s)} = K_p(1 + T_d s)$$

- proportional plus integral plus derivative controllers

$$u(t) = K_p e(t) + \frac{K_p}{T_i}\int_0^t e(t)dt + K_p T_d \frac{de(t)}{dt}$$

$$\frac{U(s)}{E(s)} = K_p(1 + \frac{1}{T_i s} + T_d s)$$

Effect of the sensor on system performance

first order sensor
over damped 2nd order sensor
under damped 2nd sensor

5-2 First order systems

Fig.5-1

RC circuit, thermal system

$$\frac{C(s)}{R(s)} = \frac{1}{Ts+1}$$

Unit step response

$$C(s) = \frac{1}{Ts+1} \cdot \frac{1}{s}$$
$$= \frac{1}{s} - \frac{T}{Ts+1} = \frac{1}{s} - \frac{1}{s+(1/T)}$$

$c(t) = 1 - e^{-t/T}$, for t >= 0

at t=T, $c(T) = 1 - e^{-1} = 0.632$
fro t >=4T , output is within 2 % error

Unit ramp response

$$C(s) = \frac{1}{Ts+1} \frac{1}{s^2}$$
$$= \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{Ts+1}$$

time response;
$c(t) = t - T + Te^{-t/T}$, for t >= 0

error signal;
$e(t) = r(t) - c(t) = T(1 - e^{-t/T})$
$e(\infty) = T$

Unit impulse response

$$C(s) = \frac{1}{Ts+1}$$
$c(t) = \frac{1}{T} e^{-t/T}$, for t>=0

5-3 Second order systems

servo system
Fig.5-5

output position: c
input position : r

$$J\ddot{c}+B\dot{c}=T$$

$$Js^2C(s)+BsC(s)=T(s)$$

$$\frac{C(s)}{T(s)}=\frac{1}{s(Js+B)}$$

$$\frac{C(s)}{R(s)}=\frac{K}{Js^2+Bs+K}=\frac{K/J}{S^2+(B/J)s+(K/J)}$$

Step response of 2nd order system

$$\frac{C(s)}{R(s)}=\frac{K}{Js^2+Bs+K}$$

$$=\frac{\frac{K}{J}}{\left[s+\frac{B}{2J}+\sqrt{(\frac{B}{2J})^2-\frac{K}{J}}\right]\left[s+\frac{B}{2J}-\sqrt{(\frac{B}{2J})^2-\frac{K}{J}}\right]}$$

Let $\frac{K}{J}=\omega_n^2$, $\frac{B}{J}=2\zeta\omega_n=2\sigma$,

$$\zeta=\frac{B}{B_c}=\frac{B}{2\sqrt{JK}}$$

$$\frac{C(s)}{R(s)}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

(1) Underdamped case ($0<\zeta<1$)

(2) Critically damped case ($\zeta = 1$)

(3) Overdamped case ($\zeta > 1$)

Definitions of transient response specifications

time domain spec.
for unit step input, at rest initially

(1) Delay time , t_d
(2) Rise time, t_r
(3) Peak time, t_p
(4) Maximum overshoot, M_p
(5) Settling time, t_s

(1) Delay time , t_d ; until half the final value
(2) Rise time, t_r ; underdamped: 0% to 100%
overdamped: 10% to 90%
(3) Peak time, t_p ; until the first peak of the overshoot
(4) Maximum (percent) overshoot, M_p ;

$$M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

indicates relative stability of the system

(5) Settling time, t_s
until to reach and stay within a range (2% or 5%) of the final value

* comments;
transient response; sufficiently fast and damped
$0.4 < \zeta < 0.8$

Second order systems and transient response specifications
(underdamped)

Rise time; t_r

$$c(t_r) = 1 - e^{-\zeta\omega_n t_r}(\cos(\omega_d t_r) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t_r)) = 1$$

$$=> \cos\omega_d t_r + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_d t_r = 0$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}(\frac{\omega_d}{-\sigma}) = \frac{\pi - \beta}{\omega_d}$$

Peak time, t_p ;

$$\frac{dc}{dt} = (\sin\omega_d t_p)\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t_p} = 0$$

$$\sin\omega_d t_p = 0$$

$$=> \omega_d t_p = 0, \pi, 2\pi, 3\pi \quad \cdots$$

$$t_p = \frac{\pi}{\omega_d}$$

1/2 cycle of the damped oscillation frequency

Maximum (percent) overshoot, M_p ;

at peak time $t_p = \frac{\pi}{\omega_d}$

$$M_p = c(t_p) - 1$$

$$= -e^{-\zeta\omega_n(\pi/\omega_d)}(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}} sin\pi)$$

$$= e^{-\sigma/\omega_d \pi} = e^{-\zeta/\sqrt{1-\zeta^2}\,\pi}$$

Settling time, t_s

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta})$$

$t_s = 4T = \frac{4}{\sigma} = \frac{4}{\zeta\omega_n}$; 2% criterion

$t_s = 3T = \frac{3}{\sigma} = \frac{3}{\zeta\omega_n}$; 5% criterion

* Servo system with velocity feedback

$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + KK_h)s + K}$$

$$\zeta = \frac{B + KK_h}{2\sqrt{KJ}}$$

Impulse response of 2nd order systems

$$C(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$0 \le \zeta \le 1$;

$$c(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin((\omega_n \sqrt{1-\zeta^2})\, t)$$

$\zeta = 1$;

$$c(t) = \omega_n^2 t e^{-\omega_n t}$$

$\zeta > 1$;

$$c(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta - \sqrt{\zeta^2-1})\omega_n t} - \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} e^{-(\zeta + \sqrt{\zeta^2-1})\omega_n t}$$

maximum overshoot for the unit impulse response (underdamped case)

$$\text{at } t = \frac{\tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n \sqrt{1-\zeta^2}}$$

$$c(t)_{\max} = \omega_n \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

5-4 Higher-Order Systems

for unit step response

Transient response

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$G(s) = \frac{p(s)}{q(s)}, \quad H(s) = \frac{n(s)}{d(s)}$$

$$\frac{C(s)}{R(s)} = \frac{p(s)d(s)}{q(s)d(s)+p(s)n(s)}$$
$$= \frac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots\ s + a_{n-1}s + a_n}, \quad (m \leqq n)$$

$$\frac{C(s)}{R(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

for a unit step input

$$C(s) = \frac{a}{s} + \sum_{i=1}^{n} \frac{a_i}{s+p_i}$$

closed loop poles;

left half s plane; the relative magnitude of the residues

=> relative importance of the component

pair of closely located poles and zeros; effective cancel each other

very far left from origin; ignore

=> approximate as lower order

real poles and comples conjugate poles

$$C(s) = \frac{K\prod_{i=1}^{m}(s+z_i)}{s\prod_{j=1}^{q}(s+p_j)\prod_{k=1}^{r}(s^2+2\zeta_k\omega_k s+\omega_k^2)}, \quad q+2r=n$$

if closed loop poles are distinct,

$$C(s) = \frac{a}{s} + \sum_{j=1}^{q}\frac{a_j}{s+p_j} + \sum_{k=1}^{r}\frac{b_k(s+\zeta_k\omega_k)+c_k\omega_k\sqrt{1-\zeta_k^2}}{s^2+2\zeta_k\omega_k s+\omega_k^2}$$

$$c(t) = a + \sum_{j=1}^{q} a_j e^{-p_j t} + \sum_{k=1}^{r} b_k e^{-\zeta_k\omega_k t}\cos\omega_k\sqrt{1-\zeta_k^2}\ t + \sum_{k=1}^{r} c_k e^{-\zeta_k\omega_k t}\sin\omega_k\sqrt{1-\zeta_k^2}\ t, \quad t>0$$

response; exponent curve + damped sinusoidal

response type ; due to closed loop poles

shape; due to closed loop zeros

poles of input $R(s)$; yields steady state response

poles of $C(s)/R(s)$; affects transient response zeros of $C(s)/R(s)$; affects on the magnitude and sign of the residues

Dominant closed loop poles

responses; determined by the ratio of the real part of the closed loop poles and the relative magnitude of the residue

ratio of the real part > 5 , no zeros nearby => poles near jw axis are dominant

dominant closed loop poles; usually complex conjugate

Stability analysis in the complex plane

poles on the right half s-plane; unstable

by the location of the closed loop poles

property of the system itself ; not from the driving function

poles on jw axis; oscillatory, if noise exist, amplitude of the oscillation increse

5-7 Routh's stability criterion

stability

Routh's stability criterion

absolute stability from the coefficient of the characteristic eq.

(1) $a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0, \quad a_n \neq 0$

(2) if any a_i = 0 or a_i<0 with at least one positive coeff.

=> root or roots with positive real part

-> unstable

(3) if all coeff. are positive,

s^n a_0 a_2 a_4 a_6 · · ·
s^{n-1} a_1 a_3 a_5 a_7 · · ·
s^{n-2} b_1 b_2 b_3 b_4 · · ·
s^{n-3} c_1 c_2 c_3 c_4 · · ·
s^{n-4} d_1 d_2 d_3 d_4 · · ·
⋮ ⋮ ⋮ ⋮ s^2 e_1 e_2 s^1
f_1 s^0 g_1

where $b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$, $b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$, $b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$, etc

$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$, $c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$, $c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$, etc

Theorem; The no. of positive real roots = no. of change of sign of the coeff. of the 1st column.

Absolute stability Necessary & sufficient condition:

all the coeff. of the charac. eq >0, and all terms of the 1st column of the array >0

Special cases;

(i) if 1st array term is zero, replace it as small positive real (ε)

$s^3 + 2s^2 + s + 2 = 0$

s^3 1 1
s^2 2 2
s^1 $0 = \epsilon$
s^0 2

ε의 윗줄과 아래줄의 부호가 같으면 pair of imaginary roots; $s = \pm j\omega$

(ii)

$$s^3 - 3s + 2 = (s-1)^2(s+2) = 0$$

s^3	1	-3
s^2	$0 = \epsilon$	2
s^1	$-3 - \frac{2}{\epsilon}$	0
s^0	2	

ε의 윗줄과 아래줄의 부호가 다르면 한번의 부호변동이 있음.
이 예제는 총 2번의 부호 변동이 있음.

(ii) if one derived row is all zero,

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

s^5	1	24	-25
s^4	2	48	-50
s^3	0	0	

Auxiliary polynomial; $P(s)$

Let $P(s) = 2s^4 + 48s^2 - 50$

$\frac{dP(s)}{ds} = 8s^3 + 96s$ => s3 row is replaced by this coeff.

s^5	1	24	-25
s^4	2	48	-50
s^3	8	96	
s^2	24	-50	
s^1	112.7	0	
s^0	-50		

=> one root with positive real part

Relative stability analysis

replace $s = \hat{s} - \sigma$

Test Routh's criterion for the $\hat{s}$ eq.

Application of Routh's stability criterion to control system analysis

Let's determine the range of K for stability

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2+s+1)(s+2)+K}$$

Characteristic eq.

$$s^4+3s^3+3s^2+2s+K=0$$

s^4	1	3	K
s^3	3	2	0
s^2	$\frac{7}{3}$	K	
s^1	$2-\frac{9}{7}K$		
s^0	K		

for stability

Routh's criterion $\quad 2-\frac{9}{7}K>0,\ \ K>0$

=> $\quad 0<K<\frac{14}{9}$

5-8 Effects of Integral and Derivative Control Actions on System Performance

Integral control action

remove offset, steady state error

oscillatory response

Proportional Control of Systems

Fig. 5-40

$$G(s) = \frac{K}{Ts+1}$$

$$E(s) = \frac{1}{1+G(s)}R(s) = \frac{1}{1+\frac{K}{Ts+1}}R(s)$$

for the unit step input $R(s) = \frac{1}{s}$

$$E(s) = \frac{Ts+1}{Ts+1+K}\frac{1}{s}$$

Steady state error;

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{Ts+1}{Ts+1+K} = \frac{1}{K+1}$$

Integral Control of Systems

Fig.5-42

$$\frac{C(s)}{R(s)} = \frac{K}{s(Ts+1)+K}$$

$$\frac{E(s)}{R(s)} = \frac{s(Ts+1)}{s(Ts+1)+K}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{s^2(Ts+1)}{Ts^2+s+K}\frac{1}{s} = 0$$

Response to torque disturbances (proportional control)
Fig.5-43

$$\frac{C(s)}{D(s)} = \frac{1}{Js^2 + bs + K_p}$$

$$\frac{E(s)}{D(s)} = -\frac{C(s)}{D(s)} = -\frac{1}{Js^2 + bs + K_p}$$

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{-s}{Js^2 + bs + K_p} \frac{T_d}{s} \\ &= -\frac{T_d}{K_p} \end{aligned}$$

Response to torque disturbances (proportional+integral control)
Fig. 5-44

$$\frac{C(s)}{D(s)} = \frac{s}{Js^3 + bs^2 + K_p s + \frac{K_p}{T_i}}$$

$$E(s) = -\frac{s}{Js^3 + bs^2 + K_p s + \frac{K_p}{T_i}} D(s)$$

if stable,

$$\begin{aligned} e_{ss} &= \lim_{s\to 0} sE(s) \\ &= \lim_{s\to 0} \frac{-s^2}{Js^3 + bs^2 + K_p s + \frac{K_p}{T_i}} \frac{1}{s} \\ &= 0 \end{aligned}$$

Derivative control action

Proportional control of systems with inertia load
Fig.5-46

$$\frac{C(s)}{R(s)} = \frac{K_p}{Js^2 + K_p}$$ => oscillate indefinitely

Proportional + derivative control of systems with inertia load
Fig. 5-47

$$\frac{C(s)}{R(s)} = \frac{K_p(1+T_d s)}{Js^2 + K_p T_d s + K_p}$$

=> two roots with negative real part

Proportional + derivative control of 2nd order systems
Fig.5-48

$$\frac{C(s)}{R(s)} = \frac{K_p + K_d s}{Js^2 + (B+K_d)s + K_p}$$

steady state error for a unit ramp input

$$e_{ss} = \frac{B}{K_p}$$

Characteristic eq.

$$Js^2 + (B+K_d)s + K_p = 0$$

damping ratio:

$$\zeta = \frac{B+K_d}{2\sqrt{K_p J}}$$

B; small, Kp ; large, Kd; large

=> $e_{ss},$ M_p small,

5-9 Steady state errors in unity feedback control system

Classification of control systems

$$G(s) = \frac{K(T_a s+1)(T_b s+1)\cdots(T_m s+1)}{s^N(T_1 s+1)(T_2 s+1)\cdots(T_p s+1)}$$

Steady state error

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$E(S) = \frac{1}{1+G(s)}R(s)$$

using final value theorem

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0}\frac{sR(s)}{1+G(s)}$$

Static position error constant K_p

for a unit step

let $K_p = \lim_{s\to 0} G(s) = G(0)$

$$e_{ss} = \lim_{s\to 0}\frac{s}{1+G(s)}\frac{1}{s} = \frac{1}{1+K_p}$$

- for type 0 ; $K_p = \lim_{s\to 0}\frac{K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = K$
- for type 1 ; $K_p = \lim_{s\to 0}\frac{K(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots} = \infty,$ for $N \geqq 1$

Static velocity error constant K_v

for a unit ramp input

let $K_v = \lim_{s\to 0} sG(s)$

$$e_{ss} = \lim_{s\to 0}\frac{s}{1+G(s)}\frac{1}{s^2} = \frac{1}{K_v}$$

- for type 0 ; $K_v = \lim_{s\to 0}\frac{sK(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = 0$
- for type 1 ; $K_v = \lim_{s\to 0}\frac{sK(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = K$
- for type 2 ; $K_v = \lim_{s\to 0}\frac{sK(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots} = \infty,$ for $N \geqq 2$

Static acceleration error constant K_a

for a unit parabolic input; $r(t) = \frac{t^2}{2}, \quad \text{for } t \geqq 0$
$= 0, \quad \text{for } t < 0$

let $K_a = \lim_{s \to 0} s^2 G(s)$

$$e_{ss} = \lim_{s \to 0} \frac{s}{1+G(s)} \frac{1}{s^3} = \frac{1}{K_a}$$

- for type 0 ; $K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{(T_1 s+1)(T_2 s+1)\cdots} = 0$
- for type 1 ; $K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{s(T_1 s+1)(T_2 s+1)\cdots} = 0$
- for type 2 ; $K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{s^2(T_1 s+1)(T_2 s+1)\cdots} = K$
- for type 3 ; $K_a = \lim_{s \to 0} \frac{s^2 K(T_a s+1)(T_b s+1)\cdots}{s^N(T_1 s+1)(T_2 s+1)\cdots} = \infty, \quad \text{for } N \geqq 3$

	Step Input r(t) =1	Ramp Input r(t) = t	Acceleration Input r(t) = 1/2 t2
Type 0 system		∞	∞
Type 1 system	0		∞
Type 2 system	0	0	

Comparison of steady state errors in open-loop control, system and closed loop control system

for open loop

K=10, $\triangle K = 1, \ \frac{\triangle K}{K} = 0.1$

$$e_{ss} = 1 - \frac{1}{K}(K + \triangle K) = 1 - 1.1 = -0.1$$

for closed loop

$$e_{ss} = \frac{1}{1+G(0)} = \frac{1}{1 + \frac{100}{K}(K + \triangle K)}$$

$$= \frac{1}{1+110} = 0.009$$