Ch. 3 Mathematical Modeling of Dynamical Systems

3-1 Introduction

Dynamics of many systems may be described in terms of differential equations.

Mathematical models;
- many different forms

  - state-space
  - transfer function

Simplicity v.s. accuracy
- simplified model
- complicated model

Linear systems
- principle of superposition <=> cause and effect is proportional

Linear time invariant system and linear time varying system
- constant coefficients differential eq.
- function of time coefficients differential eq. ; spacecraft

Nonlinear system
- principle of superposition not applied

  - saturation
  - dead zone
  - backlash
  - square law

Linearization of nonlinear system
- small signal
- around equilibrium point

3-2 Transfer function and Impulse response function

Transfer function; of linear time invariant differential equation system

Differential eq.

ration of Laplace Transform of output to that of input with initial cond. are zero.

$$a_0 y + a_1 y + \cdots + a_{n-1}\dot{y} + a_n y = b_0 x + b_1 x + \cdots + b_{m-1}\dot{x} + b_m x \qquad (n \geq m)$$

Transfer ft'n $G(s) = \dfrac{Y(s)}{X(s)} = \dfrac{b_0 s^m + b_1 s^{m-1} + \cdots + b_{m-1}s + b_m}{a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n}$

nth order system

Comments;

Linear time invariant differential system

(1) a mathematical model; operation method b.t.w. output variable v.s. input variable
(2) system property; independent of the input magnitude
(3) including units; not physical structure
(4) using T.F., output can be studied
(5) experimentally obtained T.F.

Mechanical system

ex; satellite attitude control

(1) write the differential eq.
(2) Laplace transform assuming all initial cond. are zero
(3) T.F.; ratio b.t.w. Θ to T

Convolution Integral;

$$Y(s) = G(s)X(s)$$

$$\begin{aligned} y(t) &= \int_0^t x(\tau)g(t-\tau)d\tau \\ &= \int_0^t g(\tau)x(t-\tau)d\tau \end{aligned}$$

where g(t)=0, x(t)=0 for t<0

Impulse response function

for unit impulse input

$$Y(s) = G(s)$$

impulse response function or weighting function

$$g(t) = \mathcal{L}^{-1}[G(s)]$$

3-3 Block Diagrams

composed of many component

Block diagrams;

pictorial representation; components; signal flow
easy operation

functional block; symbol for the mathematical model; input to output
signal; by arrow

main source of energy; not explicitly
many kind of block diagram for a system; point of view

Summing point;
Branch point;
Block diagram of a closed loop system;

Open loop transfer function and feedforward transfer function;

Open loop TF = $\frac{B(s)}{E(s)} = G(s)H(s)$

Feedforward TF = $\frac{C(s)}{E(s)} = G(s)$

Closed loop TF = $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

Closed loop system subjected to a disturbance

$$\frac{C_D(s)}{D(s)} = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

$$\frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1+G_1(s)G_2(s)H(s)}$$

$$C(s) = C_R(s) + C_D(s) = \frac{G_2(s)}{1+G_1(s)G_2(s)H(s)}[G_1(s)R(s)+D(s)]$$

in case; $|G_1(s)H(s)| \gg 1$ and $|G_1(s)G_2(s)H(s)| \gg 1$

disturbance rejected

$H(s)=1$; unity feedback, input=output

Procedure for drawing a block diagram

Block diagram reduction;

The product of the TF in the feedforward direction must remain the same.

The product of the TF around the loop must remain the same.

Ex 3-1)

3-4 Modeling in state space

Modern control theory; to increase accuracy
MIMO, nonlinear, time varying
time domain approach

Conventional control theory; SISO, time invariant
frequency domain approach

State, State variables, State vector, State space
state variables; smallest set of variables that determines the behaviors for $t>=t_0$

State space equations;
input variable
output variable
state variable

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_r, t) \\
\dot{x}_2 &= f_2(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_r, t) \\
\vdots &= \qquad \vdots \\
\dot{x}_n &= f_n(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_r, t)
\end{aligned}$$

$$\begin{aligned}
y_1 &= g_1(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_r, t) \\
\vdots &= \qquad \vdots \\
y_m &= g_m(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_r, t)
\end{aligned}$$

$$\boldsymbol{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \quad \boldsymbol{y}(t) = \begin{bmatrix} y_1(t) \\ \vdots \\ y_m(t) \end{bmatrix} \quad \boldsymbol{f}(x,u,t) = \begin{bmatrix} f_1(....) \\ \vdots \\ f_n(....) \end{bmatrix} \quad \boldsymbol{g}(t) = \begin{bmatrix} g_1(....) \\ \vdots \\ g_m(....) \end{bmatrix} \quad \boldsymbol{u}(t) = \begin{bmatrix} u_1(t) \\ \vdots \\ u_r(t) \end{bmatrix}$$

$$\begin{aligned}
\dot{x}(t) &= f(x, u, t) \\
y(t) &= g(x, u, t)
\end{aligned}$$

$$\begin{aligned}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{aligned}$$

Ex 3-3)

$$m\ddot{y}+b\dot{y}+ky=u$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ \frac{-k}{m} & -\frac{b}{m}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u$$

$$y=[1 \quad 0]\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Correlation between T.F. and state space eq.

$$\dot{\boldsymbol{x}} = \boldsymbol{A}x + \boldsymbol{B}u$$
$$y = \boldsymbol{C}x + Du$$

$$s\boldsymbol{X}(s) - \boldsymbol{x}(0) = \boldsymbol{AX}(s) + \boldsymbol{B}U(s)$$
$$Y(s) = \boldsymbol{CX}(s) + DU(s)$$

assume $\boldsymbol{x}(s) = 0$;

$$(s\boldsymbol{I} - \boldsymbol{A})\boldsymbol{X}(s) = \boldsymbol{B}U(s)$$
$$Y(s) = [\boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D]U(s)$$

$$G(s) = \boldsymbol{C}(s\boldsymbol{I} - \boldsymbol{A})^{-1}\boldsymbol{B} + D$$
$$= \frac{Q(s)}{|s\boldsymbol{I} - \boldsymbol{A}|}$$

Comment;

$|s\boldsymbol{I} - \boldsymbol{A}|$ == characteristic polunominal of G(s)

eigen value of $\boldsymbol{A}$ are identical to the poles of G(s)

Ex 3-4)

3-5 State Space Representation of Dynamic Systems

$$* \quad \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = u$$

state variables;

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$
$$\vdots = \vdots$$
$$x_n = \overset{(n-1)}{y}$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$y = [1 \ 0 \ \cdots \ 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$* \quad \overset{(n)}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{(n)}{u} + b_1 \overset{(n-1)}{u} + \cdots + b_{n-1}\dot{u} + b_n u$$

state variables;

$$x_1 = y - \beta_0 u$$
$$x_2 = \dot{y} - \beta_0 \dot{u} - \beta_1 u = \dot{x}_1 - \beta_1 u$$
$$x_3 = \ddot{y} - \beta_0 \ddot{u} - \beta_1 \dot{u} - \beta_2 u = \dot{x}_2 - \beta_2 u$$
$$\vdots = \vdots$$
$$x_n = \overset{(n-1)}{y} - \beta_0 \overset{(n-1)}{u} - \beta_1 \overset{(n-2)}{u} - \cdots \beta_{n-2}\dot{u} - \beta_{n-1} u = \dot{x}_{n-1} - \beta_{n-1} u$$

$$\beta_0 = b_0$$
$$\beta_1 = b_1 - a_1 \beta_0$$
$$\beta_2 = b_2 - a_1 \beta_1 - a_2 \beta_0$$
$$\vdots = \vdots$$
$$\beta_n = b_n - a_1 \beta_{n-1} - \cdots - a_{n-1} \beta_1 - a_n \beta_0$$

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$y = [1 \ 0 \ \cdots \ 0] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \beta_0 u$$

3-10 Linearization of Nonlinear Mathematical Models

utilize numerous linear system techniques
Expansion into a taylor series about the operating point
neglecting nonlinear terms

* Linear approximation of nonlinear mathematical models

(1) nonlinear system; $y(t) = f(x)$

operating point; $\bar{x}$, $\bar{y}$

$$
\begin{aligned}
y &= f(x) \\
&= f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}}(x-\bar{x}) + \frac{1}{2!}\frac{d^2 f}{dx^2}\Big|_{\bar{x}}(x-\bar{x})^2 + \cdots
\end{aligned}
$$

neglect high order terms

$$
\begin{aligned}
y &= f(\bar{x}) + \frac{df}{dx}\Big|_{\bar{x}}(x-\bar{x}) \\
&= \bar{y} + K(x-\bar{x})
\end{aligned}
$$

$$y - \bar{y} = K(x-\bar{x})$$

(2) nonlinear system; $y = f(x_1, x_2)$

Taylor series expansion;

$$
\begin{aligned}
y &= f(\overline{x_1}, \overline{x_2}) + [\frac{\partial f}{\partial x_1}(x_1 - \overline{x_1}) + \frac{\partial f}{\partial x_2}(x_2 - \overline{x_2})] \\
&+ \frac{1}{2!}[\frac{\partial^2 f}{\partial x_1^2}(x_1 - \overline{x_1})^2 + 2\frac{\partial^2 f}{\partial x_1 \partial x_2}(x_1 - \overline{x_1})(x_2 - \overline{x_2})\frac{\partial^2 f}{\partial x_2^2}(x_2 - \overline{x_2})^2] + \cdots
\end{aligned}
$$

$$y - \bar{y} = K_1(x_1 - \overline{x_1}) + K_2(x_2 - \overline{x_2})$$

$$
\begin{aligned}
\bar{y} &= f(\overline{x_1}, \overline{x_2}) \\
K_1 &= \frac{\partial f}{\partial x_1}\Big|_{\overline{x_1}, \overline{x_2}} \\
K_2 &= \frac{\partial f}{\partial x_2}\Big|_{\overline{x_1}, \overline{x_2}}
\end{aligned}
$$