Ch. 2 The Laplace Transform

2.1 Introduction

operational method for solving linear differential eq.
differential/ integral -> algebraic operation

* predicting system performance using graphical technique
* obtain transient and steady state solution simultaneously

2.2 Review of Complex variables

complex variable $s=\sigma+j\omega$

complex function $F(s)=F_x+jF_y$

magnitude; $\sqrt{F_x^2+F_y^2}$

angle; $\Theta=\tan^{-1}(F_y/F_x)$

complex conjugate; $\overline{F}(s)=F_x-j\,F_y$

- single valued function of s and uniquely determined

* analytic

analytic in a region if G(s) and all its derivatives exist

$$\frac{d}{ds}G(s)=\lim_{\Delta s\to 0}\frac{G(s+\Delta s)-G(s)}{\Delta s}=\lim_{\Delta s\to 0}\frac{\Delta G(s)}{\Delta s}$$

Cauchy-Riemann conditions

$$\frac{\partial G_x}{\partial\sigma}=\frac{\partial G_y}{\partial\omega}\quad\text{and}\quad\frac{\partial G_y}{\partial\sigma}=-\frac{\partial G_x}{\partial\omega}$$

* example

$$G(s)=\frac{1}{(s+1)}\ ,\qquad G(\sigma+j\omega)=G_x+jG_y=\frac{\sigma+1}{(\sigma+1)^2+\omega^2}+j\frac{-\omega}{(\sigma+1)^2+\omega^2}$$

is analytic in the entire s plane except at s=-1

$$\frac{d}{ds}\left(\frac{1}{s+1}\right)=-\frac{1}{(s+1)^2}$$

* ordinary point
  at which the function G(s) is analytic
* singular point
  at which the function G(s) is not analytic
* poles
  singular points at which G(s) or G′(s) approaches infinity
  simple pole
  second order pole, third order pole .....
* zeros
  at which G(s) equals zero

* example

$$G(s) = \frac{K(s+2)(s+10)}{s(s+1)(s+5)(s+15)^2}$$

poles; s=0, −1, −5, −15, −15
zeros; s=−2, −10, ∞, ∞, ∞

* Euler theorem

$$\cos\theta + j\sin\theta = e^{j\theta}$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$

2.3 Laplace Transformation

* Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_0^{\infty} f(t) e^{-st} dt$$

* Inverse Laplace transform

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds, \quad \text{for } t>0$$

c; abscissa of convergence: larger than the real part of all singular points of F(s)

- assume f(t) = 0 for t<0

* Existence of Laplace transform

F(s) exist iff Laplace integral converges

i.e. f(t) is sectionally continuous in every finite interval in the range t>0
and if it is of exponential order

- exponential order

$e^{-\sigma t}|f(t)|$ approach zero as t -> ∞ for $\sigma > \sigma_c$

∞ for $\sigma < \sigma_c$

σ_c ; abscissa of convergence

* examples

* Laplace transform for commonly used functions

- exponential function

- step function

- ramp function

- sinusoidal function

- translated function

- pulse function

- impulse function

- multiplication of f(t) by $e^{-\alpha t}$

- change of time scale

2-4 Laplace Transform Theorems

Real differentiation theorem

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right]=sF(s)-f(0)$$

$$\mathcal{L}\left[\frac{d^2}{dt^2}f(t)\right]=s^2F(s)-sf(0)-\dot{f}(0)$$

$$\mathcal{L}\left[\frac{d^n}{dt^n}f(t)\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}\dot{f}(0)-\cdots s\overset{(n-2)}{f}(0)-\overset{(n-1)}{f}(0)$$

Final value theorem

steady state behavior of f(t) as t -> infinity

$$\lim_{t\to\infty}f(t)=\lim_{s\to 0}sF(s)$$

- not applied when poles of sF(s) are at imaginary axis or right half plane

Initial value theorem

f(t) value at t=0+

$$f(0+)=\lim_{s\to\infty}sF(s)$$

- not limited as to the location of poles of sF(s)

Real integration theorem

$$\mathcal{L}\left[\int f(t)dt\right]=\frac{F(s)}{s}+\frac{f^{-1}(0)}{s}$$

where $F(s)=\mathcal{L}[f(t)],\ f^{-1}(0)=\int f(t)dt$ at $t=0$

$$\mathcal{L}\left[\int_0^t f(t)dt\right]=\frac{F(s)}{s}$$ if f(t) is exponential order

proof; since $\int_0^t f(t)dt=\int f(t)dt-f^{-1}(0)$

$$\mathcal{L}\left[\int_0^t f(t)dt\right]=\frac{F(s)}{s}+\frac{f^{-1}(0)}{s}-\frac{f^{-1}(0)}{s}=\frac{F(s)}{s}$$

Complex differentiation theorem

$$\mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$
$$\mathcal{L}[t^2 f(t)] = \frac{d^2}{ds^2}F(s)$$
$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}F(s) \qquad n = 1, 2, 3\cdots$$

Convolution integral

convolution; $f_1(t) * f_2(t) = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau = f_2(t) * f_1(t)$

if $f_1(t)$ and $f_2(t)$ is piecewise continuous and of exponential order

$$\mathcal{L}\left[\int_0^t f_1(t-\tau) f_2(\tau)\, d\tau\right] = F_1(s) F_2(s)$$

Laplace transform of product of two time function

$$\mathcal{L}[f(t)g(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(p) G(s-p)\, dp$$

c; abscissa of convergence for $F(s)$

2-5 Inverse Laplace Transform

inverse Laplace integral is complicated
use table

partial fraction expansion method for finding inverse Laplace transforms

$$F(s) = \frac{B(s)}{A(s)}$$

order of A(s) > order of B(s)

$$\begin{aligned} F(s) &= F_1(s) + F_2(s) + F_3(s) + \cdots + F_n(s) \\ \mathcal{L}^{-1}[F(s)] &= f_1(t) + f_2(t) + \cdots + f_n(t) \end{aligned}$$

when F(s) involves distinct poles only

$$\begin{aligned} F(s) = \frac{B(s)}{A(s)} &= \frac{k(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)} \qquad \text{for } m < n \\ &= \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \cdots \frac{a_n}{s+p_n} \end{aligned}$$

when F(s) involves multiple poles

$$F(s) = \frac{B(s)}{A(s)} = \frac{s^2+2s+3}{(s+1)^3}$$

$$= \frac{b_1}{s+1} + \frac{b_2}{(s+1)^2} + \frac{b_3}{(s+1)^3}$$

2-7 Solving linear time invariant differential eq.

by Laplace transform
complementary + particular solution

if initial value is zero -> d/dt ; s, d2/dt2 ; s2

(1) Laplace transform of each term
(2) Inverse Laplace transform for time solution